

On Time-Consistent Solution to Time-Inconsistent Linear-Quadratic Optimal Control of Discrete-Time Stochastic Systems *

Xun Li[†] Yuan-Hua Ni[‡] Ji-Feng Zhang[§]

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Abstract: In this paper, we investigate a class of time-inconsistent discrete-time stochastic linear-quadratic optimal control problems, whose time-consistent solutions consist of an open-loop equilibrium control and a linear feedback equilibrium strategy. The open-loop equilibrium control is defined for a given initial pair, while the linear feedback equilibrium strategy is defined for all the initial pairs. Maximum-principle-type necessary and sufficient conditions containing stationary and convexity are derived for the existence of these two time-consistent solutions, respectively. Furthermore, for the case where the system matrices are independent of the initial time, we show that the existence of the open-loop equilibrium control for a given initial pair is equivalent to the solvability of a set of nonsymmetric generalized difference Riccati equations and a set of linear difference equations. Moreover, the existence of linear feedback equilibrium strategy is equivalent to the solvability of another set of symmetric generalized difference Riccati equations.

Key words: Time-inconsistency, stochastic linear-quadratic optimal control, forward-backward stochastic difference equation

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1 Introduction

Consider the following discrete-time dynamic system

$$\begin{cases} X_{k+1} = f(k, X_k, u_k, w_k), \\ X_t = x \in \mathbb{R}^n, \quad k \in \mathbb{T}_t, \quad t \in \mathbb{T}, \end{cases} \quad (1.1)$$

where $\mathbb{T} = \{0, 1, \dots, N-1\}$ with N being a positive integer, and $\mathbb{T}_t = \{t, \dots, N-1\}$ for $t \in \mathbb{T}$. In (1.1), $\{X_k, k \in \mathbb{T}_t\}$ and $\{u_k, k \in \mathbb{T}_t\}$ with $\mathbb{T}_t = \{t, t+1, \dots, N\}$ are the state process and the control process, respectively; and $\{w_k, k \in \mathbb{T}_t\}$ is a stochastic disturbance process. Introduce the following cost functional associated with (1.1)

$$J(t, x; u) = \sum_{k=t}^{N-1} \mathbb{E}[e^{-\delta(k-t)} L(k, X_k, u_k)] + \mathbb{E}[e^{-\delta(N-t)} h(X_N)]. \quad (1.2)$$

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[†]Department of Applied Mathematics, The Hong Kong Polytechnic University, Hunghom, Kowloon, Hong Kong, P.R. China.

[‡]College of Computer and Control Engineering, Nankai University, Tianjin 300350, P. R. China

[§]Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, and the School of Mathematical Sciences, University of Chinese Academy of Sciences, P. R. China.

Let $\mathcal{U}[t, N-1]$ be a set of admissible controls. Then, the standard discrete-time stochastic optimal control problem is stated as follows.

Problem (C). For (1.1), (1.2) and the initial pair $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, find a $\bar{u} \in \mathcal{U}[t, N-1]$ such that

$$J(t, x; \bar{u}) = \inf_{u \in \mathcal{U}[t, N-1]} J(t, x; u). \quad (1.3)$$

Any $\bar{u} \in \mathcal{U}[t, N-1]$ satisfying (1.3) is called an optimal control for the initial pair (t, x) ; $\bar{X} = \{\bar{X}_k = \bar{X}(k; t, x, \bar{u}), k \in \mathbb{T}_t\}$ is called the corresponding optimal trajectory, and (\bar{X}, \bar{u}) is referred as an optimal pair.

For Problem (C), dynamic programming is a fundamental technique to find its optimal control. By studying the incremental behavior of optimal cost-to-go function as one works backward in time, the derived difference equation is termed as Bellman dynamic programming equation, which is associated with Bellman's principle of optimality. Let \bar{u} be an optimal control of Problem (C) for the initial pair (t, x) , and reconsider versions of Problem (C) along the optimal trajectory \bar{X} . For any $\tau \in \mathbb{T}_{t+1} = \{t+1, \dots, N-1\}$, Bellman's principle of optimality tells us that $\bar{u}|_{\mathbb{T}_\tau}$ (the restriction of \bar{u} on $\mathbb{T}_\tau = \{\tau, \dots, N-1\}$) is an optimal control of Problem (C) for the initial pair $(\tau, \bar{X}(\tau; t, x, \bar{u}))$. This property is also referred as the time consistency of optimal control. Time consistency ensures that one needs only to solve an optimal control problem for a given initial pair, and the obtained optimal control is also optimal along the optimal trajectory.

However, the time consistency of optimal control fails quite often in many situations. To see this, let us look at a simple example.

Example 1.1. Consider a stochastic linear-quadratic (LQ, for short) optimal control with an one-dimensional controlled system

$$\begin{cases} X_{k+1} = (X_k + u_k) + X_k w_k, \\ X_t = x \in \mathbb{R}, \quad t = 0, \quad k \in \{t, \dots, 3\} \end{cases} \quad (1.4)$$

and the cost functional

$$J(t, x; u) = \sum_{k=t}^3 \mathbb{E} \left[\frac{1}{1 + (k-t)} u_k^2 \right] + \mathbb{E} \left[\frac{2}{1 + (4-t)} X_4^2 \right]. \quad (1.5)$$

Here, $w_0, \dots, w_3 \in \mathbb{R}$, are mutually independent with properties $\mathbb{E}[w_k] = 0$, $\mathbb{E}[w_k^2] = 1$. Find a control to minimize the cost functional (1.5).

Solution. Note that $t = 0$ and

$$J(0, x; u) \geq \frac{1}{1 + (4-0)} \sum_{k=0}^3 \mathbb{E}[u_k]^2.$$

Hence, there exists a unique optimal control for the initial pair $(0, x)$,

$$\bar{u}_k^{0,x} = -W_{0,k}^{-1} P_{0,k+1} \bar{X}_k^{0,x} \equiv \bar{K}_{0,k} \bar{X}_k^{0,x}, \quad k \in \{0, 1, 2, 3\} \quad (1.6)$$

with

$$\begin{cases} P_{0,k} = 2P_{0,k+1} - P_{0,k+1} W_{0,k}^{-1} P_{0,k+1}, \\ W_{0,k} = \frac{1}{1 + (k-0)} + P_{0,k+1}, \\ P_{0,4} = \frac{2}{1 + (4-0)}, \quad k \in \{0, 1, 2, 3\} \end{cases}$$

and

$$\begin{cases} \bar{X}_{k+1}^{0,x} = (\bar{X}_k^{0,x} + \bar{u}_k^{0,x}) + \bar{X}_k^{0,x} w_k, \\ \bar{X}_0^{0,x} = x, \quad k \in \{0, 1, 2, 3\}. \end{cases}$$

Applying $\bar{u}^{0,x}$, we get $\bar{X}_1^{0,x}$. Reconsider the controlled system

$$\begin{cases} X_{k+1} = (X_k + u_k) + X_k w_k, \\ X_1 = \bar{X}_1^{0,x}, \quad k \in \{1, 2, 3\} \end{cases}$$

with the cost functional (1.5) (of $t = 1$), i.e.,

$$J(1, \bar{X}_1^{0,x}; u) = \sum_{k=1}^3 \mathbb{E} \left[\frac{1}{1+(k-1)} u_k^2 \right] + \mathbb{E} \left[\frac{2}{1+(4-1)} X_4^2 \right].$$

For such a new LQ problem (with the initial pair $(1, \bar{X}_1^{0,x})$), its optimal control is given by

$$\hat{u}_k = -W_{1,k}^{-1} P_{1,k+1} \hat{X}_k \equiv \hat{K}_{1,k} \hat{X}_k, \quad k \in \{1, 2, 3\} \quad (1.7)$$

with

$$\begin{cases} \hat{X}_{k+1} = (\hat{X}_k + \hat{u}_k) + \hat{X}_k w_k, \\ \hat{X}_1 = \bar{X}_1^{0,x}, \quad k \in \{1, 2, 3\} \end{cases}$$

and

$$\begin{cases} P_{1,k} = 2P_{1,k+1} - P_{1,k+1} W_{1,k}^{-1} P_{1,k+1}, \\ W_{1,k} = \frac{1}{1+(k-1)} + P_{1,k+1}, \\ P_{1,N} = \frac{2}{1+(4-1)}, \quad k \in \{1, 2, 3\}. \end{cases}$$

A simple calculation yields

$$\bar{K}_{0,1} = -0.6038 \text{ and } \hat{K}_{1,1} = -0.4979,$$

which imply $\bar{u}_1^{0,x} \neq \hat{u}_1$. Hence, the restriction of $\bar{u}^{0,x}$ on $\{1, 2, 3\}$ is not the optimal control for the initial pair $(1, \bar{X}_1^{0,x})$ as it is different from (1.7). Such a phenomenon is referred as the time inconsistency. \square

Both $e^{-\delta(k-t)}$ and $\frac{1}{1+(k-t)}$ in the cost functionals (1.2) and (1.5) are called the time-discounting functions. The term time discounting is broadly used “to encompass any reason for caring less about a future consequence, including factors that diminish the expected utility generated by a future consequence” [10]. $e^{-\delta(k-t)}$ is referred as the constant discounting or exponential discounting, and δ is termed as the discounting rate. As exponential function has the property of group, i.e., $e^{-\delta(k-t)} = e^{-\delta(k-\tau)} e^{-\delta(\tau-t)}$, intertemporal decision optimization problems with exponential discounting, including Problem (C), are time-consistent. Such kind of decision optimization problems is extensively studied in the communities of economics, finance, and control, etc.

Functions like $\frac{1}{1+(k-t)}$ are of hyperbolic discounting, which is well documented and often used to describe the situations with declining discounting rate. As the hyperbolic discounting function loses the property of group, the dynamic optimization problem is time-inconsistent in the sense that Bellman’s principle of optimality no longer holds. In the survey paper [10], several types of experimental evidences are presented to show the reasonableness of hyperbolic discounting other than exponential discounting; a particular example is that people always prefer the smaller-sooner reward to the larger-later reward. Furthermore, quasi-geometric discounting, mean variance utility and endogenous habit formation are several other representative examples [5, 15] that will ruin the time consistency.

Introduce the system

$$\begin{cases} X_{k+1} = f(t, k, x, X_k, u_k, w_k), \\ X_t = x \in \mathbb{R}^n, \quad k \in \mathbb{T}_t, \quad t \in \mathbb{T} \end{cases} \quad (1.8)$$

and the cost functional

$$J(t, x; u) = \sum_{k=t}^{N-1} \mathbb{E} [L(t, k, x, X_k, u_k)] + \mathbb{E} [h(t, x, X_N)]. \quad (1.9)$$

Consider the following optimal control problem.

Problem (N). For (1.8), (1.9) and the initial pair $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, find a $\bar{u} \in \mathcal{U}[t, N - 1]$ such that

$$J(t, x; \bar{u}) = \inf_{u \in \mathcal{U}[t, N - 1]} J(t, x; u),$$

where $\mathcal{U}[t, N - 1]$ is a set of admissible controls.

Different from (1.1) and (1.2), the initial time t enters explicitly into (1.8) and (1.9). This means the controlled system and the cost functional are modified at different initial times. Furthermore, the cost functional (1.9) can also be viewed as the general discounting, which includes the hyperbolic discounting, exponential discounting and quasi-geometric discounting as special cases. Therefore, Problem (N) is time-inconsistent, in general.

Due to the time inconsistency, there are two different ways to handle Problem (N). The first one is the static formulation or pre-commitment formulation. If the initial strategy is kept all the way from the initial time, then the strategy can be implemented as planned. This approach neglects the time inconsistency, and the optimal control is optimal only when viewed at the initial time. Differently, another approach addresses the time inconsistency in a dynamic manner. Instead of seeking an “optimal control”, some kinds of equilibrium solutions are dealt with. This is mainly motivated by practical applications in economics and finance, and has recently attracted considerable interest and efforts.

The explicit formulation of time inconsistency was initiated by Strotz [22] in 1955, whereas its qualitative analysis can be traced back to the work of Smith [21]. In the discrete-time case, Strotz’s idea is to tackle the time inconsistency by a lead-follower game with hierarchical structure. Particularly, controls at different time points were viewed as different selves (players), and every self integrated the policies of his successor into his own decision. By a backward procedure, the equilibrium policy (*if exists*) was obtained. Inspired by Strotz and intending to tackling practical problems in economics and finance, lots of works were concerned with time inconsistency of dynamic systems described by ordinary difference or differential equations; see, for example, [8, 9, 11, 15, 16, 18] and references therein. Unfortunately, as pointed out by Ekeland [8, 9], Strotz’s equilibrium policy typically fails to exist and is hard to prove the existence. Thus, it is of great importance to develop a general theory on time-inconsistent optimal control. This, on the one hand, can enrich the optimal control theory, and on the other hand, can provide instructive methodology to push the solvability of practical problems. Recently, this topic has attracted considerable attention from the theoretic control community; see, for example, [5, 13, 14, 24, 26, 28, 29] and references therein.

Concerned with the time-inconsistent LQ problems, we study two kinds of time-consistent equilibrium solutions, which are the open-loop equilibrium control and the closed-loop equilibrium strategy. The separate investigations of such two formulations are due to the fact that in the dynamic game theory, open-loop control distinguishes significantly from closed-loop strategy [3, 4, 23, 30]. To compare, open-loop formulation is to find an open-loop equilibrium “control”, while the “strategy” is the object of closed-loop formulation. By a strategy, we mean a decision rule that a controller uses to select a control action based on the available information set. Mathematically, a strategy is a mapping or operator on the information set. When substituting the available information into a strategy, the open-loop value or open-loop realization of this strategy is obtained. Furthermore, an open-loop equilibrium control is corresponding to a given initial pair, whereas the linear feedback equilibrium strategy is defined for all the initial pairs. For more about the terms of open-loop, closed-loop, control and strategy, we refer the readers to, for example, the monograph [3].

Strotz’s equilibrium solution [22] is essentially a closed-loop equilibrium strategy, which is further elaborately developed by Yong to the LQ optimal control [26, 29] as well as the nonlinear optimal control [28, 27]. In contrast, open-loop equilibrium control is extensively studied by Hu-Jin-Zhou [13, 14] and Yong [29]. In particular, the closed-loop formulation can be viewed as the extension of Bellman’s dynamic programming, and the corresponding equilibrium strategy (*if exists*) is derived by a backward procedure [26, 27, 28, 29]. Differently, the open-loop equilibrium control is characterized

via the maximum-principle-like methodology [13, 14].

Though the time-inconsistent optimal control has gained considerable attention, its theory is far from being mature. Concerned with the LQ problems, general necessary and sufficient conditions still do not include the existence of time-consistent equilibrium control/strategy. Hence, more elaborate efforts should be paid on such topic, and more insightful results are much desirable. In this paper, a general discrete-time time-inconsistent stochastic LQ optimal control is investigated, and no definiteness constraint is posed on the state and control weighting matrices. Such indefinite setting provides a maximal capacity to model and deal with LQ-type problems, whose study will generalize existing results to some extent. For more about standard (time-consistent) indefinite LQ problems, readers are referred to [1, 2, 7]. Under this general condition, this paper intends obtaining some neat results on the existence of time-consistent equilibrium control/strategy.

The remainder of this paper is organized as follows. In Section 2, the open-loop time-consistent equilibrium control of Problem (LQ) for a given initial pair is introduced, whose existence is characterized by some maximum-principle-type conditions. The existence of an open-loop equilibrium control for any given initial pair is then shown to be equivalent to the solvability of a set of nonsymmetric generalized difference Riccati equations (GDREs, for short) and a set of linear difference equations (LDEs, for short). In Section 3, the linear feedback time-consistent equilibrium strategy is investigated, which is defined for all the initial pairs. By also a maximum-principle-like methodology, a set of symmetric GDREs is introduced to characterize the existence of linear feedback time-consistent equilibrium strategy. Section 4 presents some comparisons between the open-loop equilibrium control and the linear feedback equilibrium strategy. Section 5 gives several examples, and Section 6 concludes the paper.

2 Open-loop time-consistent equilibrium control

Consider the following controlled stochastic difference equation (SDE, for short)

$$\begin{cases} X_{k+1}^t = A_{t,k}X_k^t + B_{t,k}u_k + (C_{t,k}X_k^t + D_{t,k}u_k)w_k, \\ X_t^t = x, \quad k \in \mathbb{T}_t, \quad t \in \mathbb{T}, \end{cases} \quad (2.1)$$

where $A_{t,k}, C_{t,k} \in \mathbb{R}^{n \times n}$, $B_{t,k}, D_{t,k} \in \mathbb{R}^{n \times m}$ are deterministic matrices; $\{X_k^t, k \in \bar{\mathbb{T}}_t\} \triangleq X^t$ and $\{u_k, k \in \mathbb{T}_t\} \triangleq u$ are the state process and the control process, respectively. In (2.1), the initial time t in these matrices and the state is to emphasize the property that the matrices and the state may change according to t . The noise $\{w_k, k \in \mathbb{T}\}$ is assumed to be a martingale difference sequence defined on a probability space (Ω, \mathcal{F}, P) with

$$\mathbb{E}[w_{k+1}|\mathcal{F}_k] = 0, \quad \mathbb{E}[(w_{k+1})^2|\mathcal{F}_k] = 1, \quad k \geq 0. \quad (2.2)$$

Here, $\mathbb{E}[\cdot|\mathcal{F}_k]$ is the conditional mathematical expectation with respect to $\mathcal{F}_k = \sigma\{x_0, w_l, l = 0, 1, \dots, k\}$ and \mathcal{F}_{-1} is understood as $\{\emptyset, \Omega\}$. The cost functional associated with system (2.1) is

$$J(t, x; u) = \sum_{k=t}^{N-1} \mathbb{E}[(X_k^t)^T Q_{t,k} X_k^t + u_k^T R_{t,k} u_k] + \mathbb{E}[(X_N^t)^T G_t X_N^t], \quad (2.3)$$

where $Q_{t,k}, R_{t,k}, k \in \mathbb{T}_t$ and G_t are deterministic symmetric matrices of appropriate dimensions. Different from [13] [26] [28] [29], we do not pose any definiteness constraint on the state and the control weight matrices. Let $L_{\mathcal{F}}^2(\mathbb{T}_t; \mathcal{H})$ be a set of \mathcal{H} -valued processes such that for any its element $\nu = \{\nu_k, k \in \mathbb{T}_t\}$ ν_k is \mathcal{F}_{k-1} -measurable and $\sum_{k=t}^{N-1} \mathbb{E}|\nu_k|^2 < \infty$. In addition, for $k \in \mathbb{T}_t$, $L_{\mathcal{F}}^2(k; \mathcal{H})$ is a set of \mathcal{H} -valued random variables such that any its element ξ is \mathcal{F}_{k-1} -measurable and $\mathbb{E}|\xi|^2 < \infty$. Throughout this paper, (t, x) is called an admissible initial time-state pair or simply an initial pair for (2.1) if $t \in \mathbb{T}$ and $x \in L_{\mathcal{F}}^2(t; \mathbb{R}^n)$. Consider the following time-inconsistent stochastic LQ problem.

Problem (LQ). For (2.1), (2.3) and the initial pair (t, x) , find a $u^* \in L^2_{\mathcal{F}}(\mathbb{T}_t; \mathbb{R}^m)$, such that

$$J(t, x; u^*) = \inf_{u \in L^2_{\mathcal{F}}(\mathbb{T}_t; \mathbb{R}^m)} J(t, x; u). \quad (2.4)$$

In this section, we investigate the open-loop equilibrium control, whose definition below is a discrete-time version of [13].

Definition 2.1. $u^{t,x,*} \in L^2_{\mathcal{F}}(\mathbb{T}_t; \mathbb{R}^m)$ is called an open-loop equilibrium control of Problem (LQ) for the initial pair (t, x) , if the following inequality holds for any $k \in \mathbb{T}_t$ and $u_k \in L^2_{\mathcal{F}}(k; \mathbb{R}^m)$

$$J(k, X_k^{t,x,*}; u^{t,x,*}|_{\mathbb{T}_k}) \leq J(k, X_k^{t,x,*}; (u_k, u^{t,x,*}|_{\mathbb{T}_{k+1}})). \quad (2.5)$$

Here, $u^{t,x,*}|_{\mathbb{T}_k}$ and $u^{t,x,*}|_{\mathbb{T}_{k+1}}$ are the restrictions of $u^{t,x,*}$ on \mathbb{T}_k and \mathbb{T}_{k+1} , respectively, and $X^{t,x,*}$ is given by

$$\begin{cases} X_{k+1}^{t,x,*} = A_{k,k} X_k^{t,x,*} + B_{k,k} u_k^{t,x,*} + (C_{k,k} X_k^{t,x,*} + D_{k,k} u_k^{t,x,*}) w_k, \\ X_t^{t,x,*} = x, \quad k \in \mathbb{T}_t. \end{cases} \quad (2.6)$$

Remark 2.2. By Definition 2.1, an open-loop equilibrium control $u^{t,x,*}$ is time-consistent along the equilibrium trajectory $X^{t,x,*}$ in the sense that for any $k \in \mathbb{T}_t$, $u^{t,x,*}|_{\mathbb{T}_k}$ is an open-loop equilibrium control for the initial pair $(k, X_k^{t,x,*})$. Furthermore, (2.5) is a local optimality condition, as the control $(u_k, u^{t,x,*}|_{\mathbb{T}_{k+1}})$ differs from $u^{t,x,*}|_{\mathbb{T}_k} = (u_k^{t,x,*}, u^{t,x,*}|_{\mathbb{T}_{k+1}})$ only at time point k .

Remark 2.3. To understand the so-called “equilibrium” in Definition 2.1, we introduce a game, termed as Problem (HG), with a hierarchical structure. The cost functional of Player k ($k = t, \dots, N-1$) is

$$J_k(u_k, u_{-k}) \triangleq J(k, X_k^k; u|_{\mathbb{T}_k}) = \sum_{\ell=k}^{N-1} \mathbb{E}[(X_{\ell}^k)^T Q_{k,\ell} X_{\ell}^k + u_{\ell}^T R_{k,\ell} u_{\ell}] + \mathbb{E}[(X_N^k)^T G_k X_N^k] \quad (2.7)$$

with $u_{-k} \triangleq \{u_t, \dots, u_{k-1}, u_{k+1}, \dots, u_{N-1}\}$. In (2.7), u_k is the action of Player k , and $\{X_{\ell}^k, \ell \in \bar{\mathbb{T}}_k\} \triangleq X^k$ is the internal state of Player k driven by all the actions $\{u_{\ell}, \ell \in \mathbb{T}_t\}$. Indeed, $\{u_t, \dots, u_{k-1}\}$ enters into X^k via its initial state

$$\begin{cases} X_{\ell+1}^k = A_{k,\ell} X_{\ell}^k + B_{k,\ell} u_{\ell} + (C_{k,\ell} X_{\ell}^k + D_{k,\ell} u_{\ell}) w_{\ell}, \\ X_k^k = X_k^{k-1}, \quad \ell \in \mathbb{T}_k \end{cases} \quad (2.8)$$

with X_k^{k-1} being the internal state of Player $k-1$ at time point k , as X_k^{k-1} is essentially a functional of $\{u_t, \dots, u_{k-1}\}$. This is why we denote $J(k, X_k^k; u|_{\mathbb{T}_k})$ as $J_k(u_k, u_{-k})$. Furthermore, in (2.8), $X_k^k = X_k^{k-1}$ indicates a forward hierarchical structure of Problem (HG): Player $k-1$ could be viewed as the leader of Player k . From (2.3), (2.5) and (2.7), we have

$$J_k(u_k^{t,x,*}, u_{-k}^{t,x,*}) \leq J_k(u_k, u_{-k}^{t,x,*}).$$

Therefore, $u^{t,x,*}$ is the Nash equilibrium of Problem (HG). Hence, in Definition 2.1, we call $u^{t,x,*}$ the equilibrium control.

The following theorem is concerned with the existence of open-loop equilibrium control.

Theorem 2.4. Given an initial pair (t, x) , the following statements are equivalent.

- (i) There exists an open-loop equilibrium control of Problem (LQ) for the initial pair (t, x) .
- (ii) There exists a control $u^{t,x,*}$ such that for any $k \in \mathbb{T}_t$, the following forward-backward stochastic difference equations (FBSΔE, for short) has a solution $(X^{k,*}, Z^{k,*})$

$$\begin{cases} X_{\ell+1}^{k,*} = A_{k,\ell} X_{\ell}^{k,*} + B_{k,\ell} u_{\ell}^{t,x,*} + (C_{k,\ell} X_{\ell}^{k,*} + D_{k,\ell} u_{\ell}^{t,x,*}) w_{\ell}, \\ Z_{\ell}^{k,*} = A_{k,\ell}^T \mathbb{E}(Z_{\ell+1}^{k,*} | \mathcal{F}_{\ell-1}) + C_{k,\ell}^T \mathbb{E}(Z_{\ell+1}^{k,*} w_{\ell} | \mathcal{F}_{\ell-1}) + Q_{k,\ell} X_{\ell}^{k,*}, \\ X_k^{k,*} = X_k^{t,x,*}, \quad Z_N^{k,*} = G_k X_N^{k,*}, \quad \ell \in \mathbb{T}_k \end{cases} \quad (2.9)$$

with the stationary condition

$$0 = R_{k,k} u_k^{t,x,*} + B_{k,k}^T \mathbb{E}(Z_{k+1}^{k,*} | \mathcal{F}_{k-1}) + D_{k,k}^T \mathbb{E}(Z_{k+1}^{k,*} w_k | \mathcal{F}_{k-1}), \quad (2.10)$$

and the convexity condition

$$\inf_{\bar{u}_k \in L_{\mathcal{F}}^2(k; \mathbb{R}^m)} \left\{ \sum_{\ell=k}^{N-1} \mathbb{E}[(Y_\ell^k)^T Q_{k,\ell} Y_\ell^k] + \mathbb{E}[\bar{u}_k^T R_{k,k} \bar{u}_k] + \mathbb{E}[(Y_N^k)^T G_k Y_N^k] \right\} \geq 0. \quad (2.11)$$

Here, Y^k is given by

$$\begin{cases} Y_{\ell+1}^k = A_{k,\ell} Y_\ell^k + C_{k,\ell} Y_\ell^k w_\ell, & \ell \in \mathbb{T}_{k+1}, \\ Y_{k+1}^k = B_{k,k} \bar{u}_k + D_{k,k} \bar{u}_k w_k, \\ Y_k^k = 0, \end{cases} \quad (2.12)$$

and $X^{t,x,*}$ is given in (2.6).

Furthermore, $u^{t,x,*}$ given in (ii) is an open-loop equilibrium control.

Proof. See Appendix A. \square

To proceed, we now recall the pseudo-inverse of a matrix. By [19], for a given matrix $M \in \mathbb{R}^{n \times m}$, there exists a unique matrix in $\mathbb{R}^{m \times n}$ denoted by M^\dagger such that

$$\begin{cases} MM^\dagger M = M, & M^\dagger MM^\dagger = M^\dagger, \\ (MM^\dagger)^T = MM^\dagger, & (M^\dagger M)^T = M^\dagger M. \end{cases} \quad (2.13)$$

This M^\dagger is called the Moore-Penrose inverse of M . The following lemma is from [1].

Lemma 2.5. *Let matrices L , M and N be given with appropriate size. Then, $LXM = N$ has a solution X if and only if $LL^\dagger NMM^\dagger = N$. Moreover, the solution of $LXM = N$ can be expressed as $X = L^\dagger NMM^\dagger + Y - L^\dagger LYMM^\dagger$, where Y is a matrix with appropriate size.*

From (2.10) and Lemma 2.5, an open-loop equilibrium control is given by

$$u_k^{t,x,*} = -R_{k,k}^\dagger [B_{k,k}^T \mathbb{E}(Z_{k+1}^{k,*} | \mathcal{F}_{k-1}) + D_{k,k}^T \mathbb{E}(Z_{k+1}^{k,*} w_k | \mathcal{F}_{k-1})], \quad k \in \mathbb{T}_t. \quad (2.14)$$

Substituting (2.14) into (2.9), we get a set of FBSΔEs

$$\begin{cases} \begin{cases} X_{\ell+1}^{k,*} = A_{k,\ell} X_\ell^{t,x,*} - B_{k,\ell} R_{\ell,\ell}^\dagger [B_{\ell,\ell}^T \mathbb{E}(Z_{\ell+1}^{\ell,*} | \mathcal{F}_{\ell-1}) + D_{\ell,\ell}^T \mathbb{E}(Z_{\ell+1}^{\ell,*} w_\ell | \mathcal{F}_{\ell-1})] \\ \quad + \{C_{k,\ell} X_\ell^{k,*} - D_{k,\ell} R_{\ell,\ell}^\dagger [B_{\ell,\ell}^T \mathbb{E}(Z_{\ell+1}^{\ell,*} | \mathcal{F}_{\ell-1}) + D_{\ell,\ell}^T \mathbb{E}(Z_{\ell+1}^{\ell,*} w_\ell | \mathcal{F}_{\ell-1})]\} w_\ell, \\ Z_\ell^{k,*} = A_{k,\ell}^T \mathbb{E}(Z_{\ell+1}^{k,*} | \mathcal{F}_{\ell-1}) + C_{k,\ell}^T \mathbb{E}(Z_{\ell+1}^{k,*} w_\ell | \mathcal{F}_{\ell-1}) + Q_{k,\ell} X_\ell^{k,*}, \\ X_t^{k,*} = X_k^{t,x,*}, \quad Z_N^{k,*} = G_k X_N^{k,*}, \quad \ell \in \mathbb{T}_{k+1}, \\ k \in \mathbb{T}_t, \end{cases} \end{cases} \quad (2.15)$$

coupled with

$$\begin{cases} X_{k+1}^{t,x,*} = A_{k,k} X_k^{t,x,*} - B_{k,k} R_{k,k}^\dagger [B_{k,k}^T \mathbb{E}(Z_{k+1}^{k,*} | \mathcal{F}_{k-1}) + D_{k,k}^T \mathbb{E}(Z_{k+1}^{k,*} w_k | \mathcal{F}_{k-1})] \\ \quad + \{C_{k,k} X_k^{t,x,*} - D_{k,k} R_{k,k}^\dagger [B_{k,k}^T \mathbb{E}(Z_{k+1}^{k,*} | \mathcal{F}_{k-1}) + D_{k,k}^T \mathbb{E}(Z_{k+1}^{k,*} w_k | \mathcal{F}_{k-1})]\} w_k, \\ X_t^{t,x,*} = x, \quad k \in \mathbb{T}_t. \end{cases}$$

To get a more convenient form of the open-loop equilibrium control, we should decouple (2.15) to obtain some Riccati-like equations. However, (2.15) is not a single FBSΔE but a set of FBSΔEs

coupled with the open-loop equilibrium state $X^{t,x,*}$. To this end, we will focus on a specific case where the system equation is

$$\begin{cases} X_{k+1} = (A_k X_k + B_k u_k) + (C_k X_k + D_k u_k) w_k, \\ X_t = x, \quad k \in \mathbb{T}_t. \end{cases} \quad (2.16)$$

Different from (2.1), the system matrices in (2.16) are assumed to be independent of the initial time t . The following result is on the equivalent characterization of the existence of open-loop equilibrium control of Problem (LQ) corresponding to (2.16) and (2.3).

Theorem 2.6. *The following statements are equivalent.*

(i) *For any initial pair $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, there exists an open-loop equilibrium control of Problem (LQ) corresponding to (2.16), (2.3) and the initial pair (t, x) .*

(ii) *The set of GDREs*

$$\begin{cases} \begin{cases} P_{t,k} = Q_{t,k} + A_k^T P_{t,k+1} A_k + C_k^T P_{t,k+1} C_k - H_{t,k}^T W_{t,k}^\dagger H_{k,k}, \\ P_{t,N} = G_t, \\ k \in \mathbb{T}_t, \\ W_{t,t} W_{t,t}^\dagger H_{t,t} - H_{t,t} = 0, \\ t \in \mathbb{T} \end{cases} \end{cases} \quad (2.17)$$

and the set of LDEs

$$\begin{cases} \begin{cases} S_{t,k} = Q_{t,k} + A_k^T S_{t,k+1} A_k + C_k^T S_{t,k+1} C_k, \\ S_{t,N} = G_t, \\ k \in \mathbb{T}_t \\ R_{t,t} + B_t^T S_{t,t+1} B_t + D_t^T S_{t,t+1} D_t \geq 0, \\ t \in \mathbb{T} \end{cases} \end{cases} \quad (2.18)$$

are solvable in the sense that the constrained conditions $W_{t,t} W_{t,t}^\dagger H_{t,t} - H_{t,t} = 0$, $R_{t,t} + B_t^T S_{t,t+1} B_t + D_t^T S_{t,t+1} D_t \geq 0$, $t \in \mathbb{T}$, are satisfied. Here,

$$\begin{cases} W_{k,k} = R_{k,k} + B_k^T P_{k,k+1} B_k + D_k^T P_{k,k+1} D_k, \\ H_{k,k} = B_k^T P_{k,k+1} A_k + D_k^T P_{k,k+1} C_k, \\ H_{t,k} = B_k^T P_{t,k+1} A_k + D_k^T P_{t,k+1} C_k, \\ k \in \mathbb{T}_t, \quad t \in \mathbb{T}. \end{cases} \quad (2.19)$$

Under any of above conditions, the open-loop equilibrium control for the initial pair (t, x) admits the feedback form

$$u_k^{t,x,*} = -W_{k,k}^\dagger H_{k,k} X_k^{t,x,*}, \quad k \in \mathbb{T}_t, \quad (2.20)$$

where

$$\begin{cases} X_{k+1}^{t,x,*} = (A_k - B_k W_{k,k}^\dagger H_{k,k}) X_k^{t,x,*} + (C_k - D_k W_{k,k}^\dagger H_{k,k}) X_k^{t,x,*} w_k, \\ X_t^{t,x,*} = x, \quad k \in \mathbb{T}_t. \end{cases} \quad (2.21)$$

Proof. (ii) \Rightarrow (i). For any given initial pair $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, let

$$\tilde{u}_k^{t,x} = -W_{k,k}^\dagger H_{k,k} \tilde{X}_k^{t,x}, \quad k \in \mathbb{T}_t, \quad (2.22)$$

where

$$\begin{cases} \tilde{X}_{k+1}^{t,x} = (A_k - B_k W_{k,k}^\dagger H_{k,k}) \tilde{X}_k^{t,x} + (C_k - D_k W_{k,k}^\dagger H_{k,k}) \tilde{X}_k^{t,x} w_k, \\ \tilde{X}_t^{t,x} = x, \quad k \in \mathbb{T}_t. \end{cases} \quad (2.23)$$

Then, the following FBSΔE

$$\begin{cases} \tilde{X}_{\ell+1}^k = A_\ell \tilde{X}_\ell^k + B_\ell \tilde{u}_\ell^{t,x} + \{C_\ell \tilde{X}_\ell^k + D_\ell \tilde{u}_\ell^{t,x}\} w_\ell, \\ \tilde{Z}_\ell^k = A_\ell^T \mathbb{E}(\tilde{Z}_{\ell+1}^k | \mathcal{F}_{\ell-1}) + C_\ell^T \mathbb{E}(\tilde{Z}_{\ell+1}^k w_\ell | \mathcal{F}_{\ell-1}) + Q_{k,\ell} \tilde{X}_\ell^k, \\ \tilde{X}_k^k = \tilde{X}_k^{t,x}, \quad \tilde{Z}_N^k = G_k \tilde{X}_N^k, \\ \ell \in \mathbb{T}_k \end{cases} \quad (2.24)$$

has a solution $(\tilde{X}^k, \tilde{Z}^k)$, as the backward state \tilde{Z}^k does not appear in the forward SΔE. Comparing the forward SΔE of (2.24) (by substituting $\tilde{u}^{t,x}$) with (2.23), we have

$$\tilde{X}_\ell^k = \tilde{X}_\ell^{t,x}, \quad \ell \in \mathbb{T}_k.$$

To apply Theorem 2.4, we validate the stationary condition. Noting $\tilde{Z}_N^k = G_N \tilde{X}_N^k$, we have

$$\begin{aligned} \tilde{Z}_{N-1}^k &= (A_{N-1}^T P_{k,N} A_{N-1} + C_{N-1}^T P_{k,N} C_{N-1} + Q_{k,N-1}) \tilde{X}_{N-1}^k \\ &\quad + (A_{N-1}^T P_{k,N} B_{N-1} + C_{N-1}^T P_{k,N} D_{N-1}) \tilde{u}_{N-1}^{t,x} \\ &= (A_{N-1}^T P_{k,N} A_{N-1} + C_{N-1}^T P_{k,N} C_{N-1} + Q_{k,N-1} - H_{k,N-1}^T W_{N-1,N-1}^\dagger H_{N-1,N-1}) \tilde{X}_{N-1}^k \\ &= P_{k,N-1} \tilde{X}_{N-1}^k. \end{aligned} \quad (2.25)$$

By some backward induction, we get

$$\tilde{Z}_\ell^k = P_{k,\ell} \tilde{X}_\ell^k, \quad \ell \in \mathbb{T}_k, \quad k \in \mathbb{T}_t,$$

which implies

$$\begin{aligned} &R_{k,k} \tilde{u}_k^{t,x} + B_k^T \mathbb{E}(\tilde{Z}_{k+1}^k | \mathcal{F}_{k-1}) + D_k^T \mathbb{E}(\tilde{Z}_{k+1}^k w_k | \mathcal{F}_{k-1}) \\ &= (R_{k,k} + B_k^T P_{k,k+1} B_k + D_k^T P_{k,k+1} D_k) \tilde{u}_k^{t,x} + (B_k^T P_{k,k+1} A_k + D_k^T P_{k,k+1} C_k) \tilde{X}_k^{t,x} \\ &= W_{k,k} \tilde{u}_k^{t,x} + H_{k,k} \tilde{X}_k^{t,x} \\ &= 0. \end{aligned}$$

The last equality follows from the solvability of (2.17). Therefore, the stationary condition holds. Furthermore, corresponding to (2.12) and (2.16), let

$$\begin{cases} Y_{\ell+1} = A_\ell Y_\ell + C_\ell Y_\ell w_\ell, & \ell \in \mathbb{T}_{k+1}, \\ Y_{k+1} = B_k \bar{u}_k + D_k \bar{u}_k w_k, \\ Y_k = 0. \end{cases}$$

Then, by some simple calculations, we have for any $\bar{u}_k \in L_{\mathcal{F}}^2(k; \mathbb{R}^m)$

$$\begin{aligned} &\mathbb{E}[\bar{u}_k^T R_{k,k} \bar{u}_k] + \mathbb{E}[(Y)_N^T G_k Y_N] + \sum_{\ell=k}^{N-1} \mathbb{E}[(Y_\ell)^T Q_{k,\ell} Y_\ell] \\ &= \sum_{\ell=k}^{N-1} \mathbb{E}[(Y_{\ell+1})^T S_{k,\ell+1} Y_{\ell+1} - (Y_\ell)^T (S_{k,\ell} - Q_{k,\ell}) Y_\ell] + \mathbb{E}[\bar{u}_k^T R_{k,k} \bar{u}_k] \\ &= \mathbb{E}[\bar{u}_k^T (R_{k,k} + B_k^T S_{k,k+1} B_k + D_k^T S_{k,k+1} D_k) \bar{u}_k] \\ &\geq 0, \end{aligned} \quad (2.26)$$

where the inequality is from the solvability of (2.18). Therefore, the convexity condition holds. By Theorem 2.4, the pair $(\tilde{X}^{t,x}, \tilde{u}^{t,x})$ given in (2.22)-(2.23) is an open-loop equilibrium pair of Problem (LQ) for (2.16), (2.3) and (t, x) .

(i) \Rightarrow (ii). For any given initial pair $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, let $(X^{t,x,*}, u^{t,x,*})$ be an open-loop equilibrium pair of Problem (LQ) corresponding to (2.16) and (2.3). In this case, for any $k \in \mathbb{T}_t$, (2.9) (2.10) become

$$\begin{cases} X_{\ell+1}^{t,x,*} = A_\ell X_\ell^{t,x,*} + B_\ell u_\ell^{t,x,*} + (C_\ell X_\ell^{t,x,*} + D_\ell u_\ell^{t,x,*}) w_\ell, \\ Z_\ell^{k,*} = Q_{k,\ell} X_k^{t,x,*} + A_\ell^T \mathbb{E}(Z_{\ell+1}^{k,*} | \mathcal{F}_{\ell-1}) + C_\ell^T \mathbb{E}(Z_{\ell+1}^{k,*} w_\ell | \mathcal{F}_{\ell-1}), \\ X_k^{t,x,*} = X_k^{t,x,*}, \quad Z_N^{k,*} = G_k X_N^{t,x,*}, \\ \ell \in \mathbb{T}_k, \end{cases} \quad (2.27)$$

and

$$0 = R_{k,k} u_k^{t,x,*} + B_k^T \mathbb{E}(Z_{k+1}^{k,*} | \mathcal{F}_{k-1}) + D_k^T \mathbb{E}(Z_{k+1}^{k,*} w_k | \mathcal{F}_{k-1}). \quad (2.28)$$

Letting $t = N - 1$, we have from (2.27) and (2.28)

$$\begin{aligned} 0 &= R_{N-1,N-1} u_{N-1}^{N-1,x,*} + B_{N-1}^T \mathbb{E}(Z_N^{N-1,*} | \mathcal{F}_{N-2}) + D_{N-1}^T \mathbb{E}(Z_N^{N-1,*} w_{N-1} | \mathcal{F}_{N-2}) \\ &= (R_{N-1,N-1} + B_{N-1}^T G_{N-1} B_{N-1} + D_{N-1}^T G_{N-1} D_{N-1}) u_{N-1}^{N-1,x,*} \\ &\quad + (B_{N-1}^T G_{N-1} A_{N-1} + D_{N-1}^T G_{N-1} C_{N-1}) X_{N-1}^{N-1,x,*} \\ &\triangleq W_{N-1,N-1} u_{N-1}^{N-1,x,*} + H_{N-1,N-1} X_{N-1}^{N-1,x,*}, \end{aligned} \quad (2.29)$$

where $W_{N-1,N-1}, H_{N-1,N-1}$ are defined in (2.19) with $P_{N-1,N} = G_{N-1}$. As x in (2.29) can be taken arbitrarily, we have from Lemma 2.5

$$W_{N-1,N-1} W_{N-1,N-1}^\dagger H_{N-1,N-1} - H_{N-1,N-1} = 0, \quad (2.30)$$

and

$$u_{N-1}^{N-1,x,*} = -W_{N-1,N-1}^\dagger H_{N-1,N-1} x.$$

Assume that we have derived the GDREs over the time period $\mathbb{T}_{t+1} = \{t+1, \dots, N-1\}$, namely,

$$\begin{cases} \begin{cases} P_{k,\ell} = Q_{k,\ell} + A_\ell^T P_{k,\ell+1} A_\ell + C_\ell^T P_{k,\ell+1} C_\ell - H_{k,\ell}^T W_{\ell,\ell}^\dagger H_{k,\ell}, \\ P_{k,N} = G_k, \\ \ell \in \mathbb{T}_k, \end{cases} \\ W_{k,k} W_{k,k}^\dagger H_{k,k} - H_{k,k} = 0, \\ k \in \mathbb{T}_{t+1} \end{cases} \quad (2.31)$$

is solvable. To prove the solvability of the GDREs over \mathbb{T}_t , it is necessary to prove that the GDRE associated with t

$$\begin{cases} \begin{cases} P_{t,k} = Q_{t,k} + A_k^T P_{t,k+1} A_k + C_k^T P_{t,k+1} C_k - H_{t,k}^T W_{k,k}^\dagger H_{t,k}, \\ P_{t,N} = G_t, \\ k \in \mathbb{T}_t, \end{cases} \\ W_{t,t} W_{t,t}^\dagger H_{t,t} - H_{t,t} = 0 \end{cases} \quad (2.32)$$

is solvable. Let $x \in \mathbb{R}^n$ and consider Problem (LQ) corresponding to (2.16), (2.3) and the initial pair (t, x) . Similarly to (2.29), we have for $k = N - 1$

$$0 = W_{N-1,N-1} u_{N-1}^{t,x,*} + H_{N-1,N-1} X_{N-1}^{t,x,*}, \quad (2.33)$$

which together with the solvability of (2.31) implies

$$u_{N-1}^{t,x,*} = -W_{N-1,N-1}^\dagger H_{N-1,N-1} X_{N-1}^{t,x,*}. \quad (2.34)$$

Similar to (2.25), we have

$$\begin{aligned}
Z_{N-1}^{N-2,*} &= \left(A_{N-1}^T G_{N-2} A_{N-1} + C_{N-1}^T G_{N-2} C_{N-1} + Q_{N-2,N-1} \right. \\
&\quad \left. - H_{N-2,N-1}^T W_{N-1,N-1}^\dagger H_{N-1,N-1} \right) X_{N-1}^{t,x,*} \\
&= P_{N-2,N-1} X_{N-1}^{t,x,*}.
\end{aligned} \tag{2.35}$$

Furthermore, from (2.27) and (2.28) we have

$$\begin{aligned}
0 &= R_{N-2,N-2} u_{N-2}^{t,x,*} + B_{N-2}^T \mathbb{E}(Z_{N-1}^{N-2,*} | \mathcal{F}_{N-3}) + D_{N-2}^T \mathbb{E}(Z_{N-1}^{N-2,*} w_{N-2} | \mathcal{F}_{N-3}) \\
&= (R_{N-2,N-2} + B_{N-2}^T P_{N-2,N-1} B_{N-2} + D_{N-2}^T P_{N-2,N-1} D_{N-2}) u_{N-2}^{t,x,*} \\
&\quad + (B_{N-2}^T P_{N-2,N-1} A_{N-2} + D_{N-2}^T P_{N-2,N-1} C_{N-2}) X_{N-2}^{t,x,*} \\
&= W_{N-2,N-2} u_{N-2}^{t,x,*} + H_{N-2,N-2} X_{N-2}^{t,x,*}.
\end{aligned}$$

By the solvability of (2.31) and Lemma 2.5, we have

$$u_{N-2}^{t,x,*} = -W_{N-2,N-2}^\dagger H_{N-2,N-2} X_{N-2}^{t,x,*}. \tag{2.36}$$

Furthermore,

$$\begin{aligned}
Z_{N-1}^{N-3,*} &= \left\{ Q_{N-3,N-1} + A_{N-1}^T G_{N-3} A_{N-1} + C_{N-1}^T G_{N-3} C_{N-1} \right. \\
&\quad \left. - H_{N-3,N-1}^T W_{N-1,N-1}^\dagger H_{N-1,N-1} \right\} X_{N-1}^{t,x,*} \\
&= P_{N-3,N-1} X_{N-1}^{t,x,*},
\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}
Z_{N-2}^{N-3,*} &= Q_{N-3,N-2} X_{N-2}^{t,x,*} + A_{N-2}^T \mathbb{E}(Z_{N-1}^{N-3,*} | \mathcal{F}_{N-3}) + C_{N-2}^T \mathbb{E}(Z_{N-1}^{N-3,*} w_{N-2} | \mathcal{F}_{N-3}) \\
&= (Q_{N-3,N-2} + A_{N-2}^T P_{N-3,N-1} A_{N-2} + C_{N-2}^T P_{N-3,N-1} C_{N-1}) X_{N-2}^{t,x,*} \\
&\quad + (A_{N-2}^T P_{N-3,N-1} B_{N-2} + C_{N-2}^T P_{N-3,N-1} D_{N-2}) u_{N-2}^{t,x,*} \\
&= (Q_{N-3,N-2} + A_{N-2}^T P_{N-3,N-1} A_{N-2} + C_{N-2}^T P_{N-3,N-1} C_{N-1} \\
&\quad - H_{N-3,N-2}^T W_{N-2,N-2}^\dagger H_{N-2,N-2}) X_{N-2}^{t,x,*} \\
&= P_{N-3,N-2} X_{N-2}^{t,x,*}.
\end{aligned} \tag{2.38}$$

From (2.34)-(2.38), we have the following deduction

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} u_\ell^{t,x,*} = -W_{\ell,\ell}^\dagger H_{\ell,\ell} X_\ell^{t,x,*}, \\ Z_\ell^{k-1,*} = P_{k-1,\ell} X_\ell^{t,x,*}, \\ \ell \in \mathbb{T}_k, \\ k \in \mathbb{T}_{t+1}. \end{array} \right. \end{array} \right. \tag{2.39}$$

To extend (2.39) to the case of including $k = t$, we have from (2.27) and (2.28)

$$\begin{aligned}
0 &= R_{t,t} u_t^{t,x,*} + B_t^T \mathbb{E}(Z_{t+1}^{t,*} | \mathcal{F}_{t-1}) + D_t^T \mathbb{E}(Z_{t+1}^{t,*} w_t | \mathcal{F}_{t-1}) \\
&= (R_{t,t} + B_t^T P_{t,t+1} B_t + D_t^T P_{t,t+1} D_t) u_t^{t,x,*} + (B_t^T P_{t,t+1} A_t + D_t^T P_{t,t+1} C_t) x \\
&= W_{t,t} u_t^{t,x,*} + H_{t,t} x.
\end{aligned}$$

As x can be arbitrarily taken, from Lemma 2.5 we have

$$W_{t,t} W_{t,t}^\dagger H_{t,t} - H_{t,t} = 0, \tag{2.40}$$

and

$$u_t^{t,x,*} = -W_{t,t}^\dagger H_{t,t} x.$$

By (2.40), the GDRE (2.32) associated with t is solvable. Thus, GDREs over \mathbb{T}_t are solvable. By the method of induction, we have the solvability of the set of GDREs (2.17).

To conclude the proof, we need to show the solvability of (2.18). From (2.11) and similar to (2.26), we have

$$0 \leq \inf_{\bar{u}_k \in L^2_{\mathcal{F}}(k; \mathbb{R}^m)} \mathbb{E}[\bar{u}_k^T (R_{k,k} + B_k^T S_{k,k+1} B_k + D_k^T S_{k,k+1} D_k) \bar{u}_k], \quad k \in \mathbb{T}.$$

Hence,

$$R_{k,k} + B_k^T S_{k,k+1} B_k + D_k^T S_{k,k+1} D_k \geq 0, \quad k \in \mathbb{T},$$

and (2.18) is solvable. \square

Remark 2.7. In (2.17), the GDREs are coupled via $\{W_{k,k}^\dagger H_{k,k}, k \in \mathbb{T}\}$. As for $k \neq t$ $H_{k,k}$ is generally not equal to $H_{t,k}$, $P_{t,k}, k \in \mathbb{T}_t, t \in \mathbb{T}$, are nonsymmetric. On the contrary, the LDEs in (2.18) are decoupled. Hence, $S_{t,k}, k \in \mathbb{T}_t, t \in \mathbb{T}$, are all symmetric as $G_t, t \in \mathbb{T}$, are symmetric. Interestingly, there is no definite constraint on matrices associated with (2.17), while the definite constraint is posed through (2.18). This is different from the standard indefinite stochastic LQ optimal control. Concerned with the reasons, the definite constraints in (2.18) are equivalent to the convexity conditions, while the constraints in (2.17) are associated with the stationary conditions.

Let us further assume that G_t and $Q_{t,k}(k \in \mathbb{T}_t)$ are all independent of t . Then, (2.17) and (2.18) become

$$\begin{cases} P_k = Q_k + A_k^T P_{k+1} A_k + C_k^T P_{k+1} C_k - H_k^T W_k^\dagger H_k, \\ P_N = G, \\ W_k W_k^\dagger H_k - H_k = 0, \\ k \in \mathbb{T}, \end{cases} \quad (2.41)$$

and

$$\begin{cases} S_k = Q_k + A_k^T S_{k+1} A_k + C_k^T S_{k+1} C_k, \\ S_N = G, \\ R_{k,k} + B_k^T S_{k+1} B_k + D_k^T S_{k+1} D_k \geq 0, \\ k \in \mathbb{T}, \end{cases} \quad (2.42)$$

where

$$\begin{cases} W_k = R_{k,k} + B_k^T P_{k+1} B_k + D_k^T P_{k+1} D_k, \\ H_k = B_k^T P_{k+1} A_k + D_k^T P_{k+1} C_k, \\ k \in \mathbb{T}. \end{cases}$$

In this case, $P_k, k \in \mathbb{T}$, are all symmetric. Review the standard GDRE [1]

$$\begin{cases} P_k = Q_k + A_k^T P_{k+1} A_k + C_k^T P_{k+1} C_k - H_k^T W_k^\dagger H_k, \\ P_N = G, \\ W_k W_k^\dagger H_k - H_k = 0, \\ W_k \geq 0, \\ k \in \mathbb{T}. \end{cases} \quad (2.43)$$

Though the GDRE (2.41) and the LDE (2.42) are different from the GDRE (2.43), we claim that (2.41) and (2.42) are both solvable for the case with $Q_k \geq 0, G \geq 0$ and $R_{k,k} \geq 0, k \in \mathbb{T}$. However, the condition that ensure the solvability of (2.17) is hard to obtain (even for the definite case) due to its nonsymmetric structure. At the present time, we therefore need to validate the solvability of (2.17) case by case. In the future, we shall study the condition that ensure the solvability of (2.17), and focus on more general cases other than the case of (2.16).

3 Linear feedback time-consistent equilibrium strategy

In this section, we investigate a kind of closed-loop equilibrium solution, which focuses on the time-consistency of the strategy. Here, a strategy means a decision rule that a controller uses to select her control action, based on available information set. Mathematically, a strategy is a measurable mapping on the information set. When we substitute the available information into a strategy, the open-loop realization or open-loop value of this strategy is obtained.

Definition 3.1. $\Phi = \{\Phi_0, \dots, \Phi_{N-1}\}$ with $\Phi_t \in \mathbb{R}^{m \times n}, t \in \mathbb{T}$, is called a linear feedback equilibrium strategy of Problem (LQ), if the following inequality holds for any initial pair $(t, x) \in \mathbb{T} \times \mathbb{R}^n, k \in \mathbb{T}_t$ and $u_k \in L^2_{\mathcal{F}}(k; \mathbb{R}^m)$

$$J(k, X_k^{t,x,*}; (\Phi X^{k,\Phi})|_{\mathbb{T}_k}) \leq J(k, X_k^{t,x,*}; (u_k, (\Phi X^{k,u_k,\Phi})|_{\mathbb{T}_{k+1}})). \quad (3.1)$$

Here, $X^{t,x,*} = \{X_k^{t,x,*}, k \in \mathbb{T}_t\}$, $X^{k,\Phi} = \{X_\ell^{k,\Phi}, \ell \in \mathbb{T}_k\}$ and $X^{k,u_k,\Phi} = \{X_\ell^{k,u_k,\Phi}, \ell \in \mathbb{T}_k\}$ are given, respectively, by

$$\begin{cases} X_{k+1}^{t,x,*} = (A_{k,k} + B_{k,k}\Phi_k)X_k^{t,x,*} + (C_{k,k} + D_{k,k}\Phi_k)X_k^{t,x,*}w_k, \\ X_t^{t,x,*} = x, \quad k \in \mathbb{T}_t, \end{cases} \quad (3.2)$$

$$\begin{cases} X_{\ell+1}^{k,\Phi} = (A_{k,\ell} + B_{k,\ell}\Phi_\ell)X_\ell^{k,\Phi} + (C_{k,\ell} + D_{k,\ell}\Phi_\ell)X_\ell^{k,\Phi}w_\ell, \\ X_k^{k,\Phi} = X_k^{t,x,*}, \quad \ell \in \mathbb{T}_k, \end{cases} \quad (3.3)$$

$$\begin{cases} X_{\ell+1}^{k,u_k,\Phi} = (A_{k,\ell} + B_{k,\ell}\Phi_\ell)X_\ell^{k,u_k,\Phi} + (C_{k,\ell} + D_{k,\ell}\Phi_\ell)X_\ell^{k,u_k,\Phi}w_\ell, \quad \ell \in \mathbb{T}_{k+1}, \\ X_{k+1}^{k,u_k,\Phi} = A_{k,k}X_k^{k,u_k,\Phi} + B_{k,k}u_k + (C_{k,k}X_k^{k,u_k,\Phi} + D_{k,k}u_k)w_k, \\ X_k^{k,u_k,\Phi} = X_k^{t,x,*} \end{cases} \quad (3.4)$$

with

$$\begin{aligned} (\Phi X^{k,\Phi})|_{\mathbb{T}_k} &= (\Phi_k X_k^{k,\Phi}, \dots, \Phi_{N-1} X_{N-1}^{k,\Phi}), \\ (\Phi X^{k,u_k,\Phi})|_{\mathbb{T}_{k+1}} &= (\Phi_{k+1} X_{k+1}^{k,u_k,\Phi}, \dots, \Phi_{N-1} X_{N-1}^{k,u_k,\Phi}). \end{aligned}$$

From Definition 3.1, we know that $\Phi|_{\mathbb{T}_t}$ (the restriction of Φ on \mathbb{T}_t) is a linear feedback equilibrium strategy. Hence, Φ is time-consistent. The following theorem is concerned with the existence of linear feedback equilibrium strategy, which is parallel to Theorem 2.4.

Theorem 3.2. *The following statements are equivalent.*

- (i) *There exists a linear feedback equilibrium strategy of Problem (LQ).*
- (ii) *There exists a $\Phi = \{\Phi_0, \dots, \Phi_{N-1}\}$ with $\Phi_t \in \mathbb{R}^{m \times n}, t \in \mathbb{T}$, such that for any initial pair $(t, x) \in \mathbb{T} \times \mathbb{R}^n$ and $k \in \mathbb{T}_t$, the following FBSΔE has a solution $(X^{k,\Phi}, Z^{k,\Phi})$*

$$\begin{cases} X_{\ell+1}^{k,\Phi} = (A_{k,\ell} + B_{k,\ell}\Phi_\ell)X_\ell^{k,\Phi} + (C_{k,\ell} + D_{k,\ell}\Phi_\ell)X_\ell^{k,\Phi}w_\ell, \\ Z_\ell^{k,\Phi} = (A_{k,\ell} + B_{k,\ell}\Phi_\ell)^T \mathbb{E}(Z_{\ell+1}^{k,\Phi} | \mathcal{F}_{\ell-1}) + (C_{k,\ell} + D_{k,\ell}\Phi_\ell)^T \mathbb{E}(Z_{\ell+1}^{k,\Phi} w_\ell | \mathcal{F}_{\ell-1}) \\ \quad + (\Phi_\ell^T R_{k,\ell} \Phi_\ell + Q_{k,\ell})X_\ell^{k,\Phi}, \\ X_k^{k,\Phi} = X_k^{t,x,*}, \quad Z_N^{k,\Phi} = G_k X_N^{k,\Phi}, \quad \ell \in \mathbb{T}_k \end{cases} \quad (3.5)$$

with the stationary condition

$$0 = R_{k,k} \Phi_k X_k^{k,\Phi} + B_{k,k}^T \mathbb{E}(Z_{k+1}^{k,\Phi} | \mathcal{F}_{k-1}) + D_{k,k}^T \mathbb{E}(Z_{k+1}^{k,\Phi} w_k | \mathcal{F}_{k-1}), \quad k \in \mathbb{T}_t, \quad (3.6)$$

and the convexity condition

$$\inf_{\bar{u}_k \in L_{\mathcal{F}}^2(k; \mathbb{R}^m)} \left\{ \mathbb{E}[\bar{u}_k^T R_{k,k} \bar{u}_k] + \sum_{\ell=k}^{N-1} \mathbb{E}[(Y_\ell^{k,\bar{u}_k,\Phi})^T (Q_{k,\ell} + \Phi_\ell^T R_{k,\ell} \Phi_\ell) Y_\ell^{k,\bar{u}_k,\Phi}] \right. \\ \left. + \mathbb{E}[(Y_N^{k,\bar{u}_k,\Phi})^T G_N Y_N^{k,\bar{u}_k,\Phi}] \right\} \geq 0. \quad (3.7)$$

In (3.5) (3.7), $X^{t,x,*}, Y^{k,\bar{u}_k,\Phi}$ are given by (3.2) and

$$\begin{cases} Y_{\ell+1}^{k,\bar{u}_k,\Phi} = (A_{k,\ell} + B_{k,\ell} \Phi_\ell) Y_\ell^{k,\bar{u}_k,\Phi} + (C_{k,\ell} + D_{k,\ell} \Phi_\ell) Y_\ell^{k,\bar{u}_k,\Phi} w_\ell, & \ell \in \mathbb{T}_{k+1}, \\ Y_{k+1}^{k,\bar{u}_k,\Phi} = B_{k,k} \bar{u}_k + D_{k,k} \bar{u}_k w_k, \\ Y_k^{k,\bar{u}_k,\Phi} = 0. \end{cases} \quad (3.8)$$

Proof. See Appendix B. \square

Based on Theorem 3.2, the relationship between the existence of linear feedback equilibrium strategy and the solvability of a set of difference equations is established. It is stated in the following theorem.

Theorem 3.3. *The following statements are equivalent.*

- (i) Problem (LQ) admits a linear feedback equilibrium strategy.
- (ii) The following set of equations

$$\begin{cases} \begin{cases} \tilde{P}_{t,k} = Q_{t,k} + \Phi_k^T R_{t,k} \Phi_k + (A_{t,k} + B_{t,k} \Phi_k)^T \tilde{P}_{t,k+1} (A_{t,k} + B_{t,k} \Phi_k) \\ \quad + (C_{t,k} + D_{t,k} \Phi_k)^T \tilde{P}_{t,k+1} (C_{t,k} + D_{t,k} \Phi_k) \\ \tilde{P}_{t,N} = G_t, \\ k \in \mathbb{T}_t, \\ \tilde{W}_{t,t} \tilde{W}_{t,t}^\dagger \tilde{H}_{t,t} - \tilde{H}_{t,t} = 0, \\ \tilde{W}_{t,t} \geq 0, \\ t \in \mathbb{T} \end{cases} \end{cases} \quad (3.9)$$

is solvable in the sense that the constrained conditions $\tilde{W}_{t,t} \tilde{W}_{t,t}^\dagger \tilde{H}_{t,t} - \tilde{H}_{t,t} = 0$ and $\tilde{W}_{t,t} \geq 0, \forall t \in \mathbb{T}$, are satisfied. Here,

$$\begin{cases} \tilde{W}_{t,t} = R_{t,t} + B_{t,t}^T \tilde{P}_{t,t+1} B_{t,t} + D_{t,t}^T \tilde{P}_{t,t+1} D_{t,t}, \\ \tilde{H}_{t,t} = B_{t,t}^T \tilde{P}_{t,t+1} A_{t,t} + D_{t,t}^T \tilde{P}_{t,t+1} C_{t,t}, \\ \Phi_t = -\tilde{W}_{t,t}^\dagger \tilde{H}_{t,t}, \\ t \in \mathbb{T}. \end{cases} \quad (3.10)$$

Under any of above conditions, Φ given in (3.10) is a linear feedback equilibrium strategy.

Proof. (i) \Rightarrow (ii). Let Φ be a linear feedback equilibrium strategy. Then, for any initial pair (t, x) , (3.5) admits a solution. Let $k = t$ in (3.5) and (3.6); noting $Z_N^{t,\Phi} = G_t X_N^{t,\Phi}$ and by results in [17], one can get

$$Z_k^{t,\Phi} = \tilde{P}_{t,k} X_k^{t,\Phi}, \quad k \in \mathbb{T}_t, \quad (3.11)$$

where $\tilde{P}_{t,k}, k \in \mathbb{T}_t$, are deterministic matrices and determined below. Then, from (3.6) and Lemma 2.5, we have

$$0 = R_{t,t} \Phi_t X_t^{t,\Phi} + B_{t,t}^T \mathbb{E}(\tilde{P}_{t,t+1} X_{t+1}^{t,\Phi} | \mathcal{F}_{t-1}) + D_{t,t}^T \mathbb{E}(\tilde{P}_{t,t+1} X_{t+1}^{t,\Phi} w_t | \mathcal{F}_{t-1})$$

$$= \left[R_{t,t} \Phi_t + B_{t,t}^T \tilde{P}_{t,t+1} (A_{t,t} + B_{t,t} \Phi_t) + D_{t,t}^T \tilde{P}_{t,t+1} (C_{t,t} + D_{t,t} \Phi_t) \right] X_t^{t,\Phi}.$$

As $x = X_t^{t,\Phi}$ can be arbitrarily selected, we have

$$0 = R_{t,t} \Phi_t + B_{t,t}^T \tilde{P}_{t,t+1} (A_{t,t} + B_{t,t} \Phi_t) + D_{t,t}^T \tilde{P}_{t,t+1} (C_{t,t} + D_{t,t} \Phi_t).$$

From Lemma 2.5, it follows that

$$\Phi_t = -\tilde{W}_{t,t}^\dagger \tilde{H}_{t,t},$$

and

$$\tilde{W}_{t,t} \tilde{W}_{t,t}^\dagger \tilde{H}_{t,t} - \tilde{H}_{t,t} = 0.$$

Note that

$$\begin{aligned} Z_k^{t,\Phi} = & \left\{ Q_{t,k} + \Phi_k^T R_{t,k} \Phi_k + (A_{t,k} + B_{t,k} \Phi_k)^T \tilde{P}_{t,k+1} (A_{t,k} + B_{t,k} \Phi_k) \right. \\ & \left. + (C_{t,k} + D_{t,k} \Phi_k)^T \tilde{P}_{t,k+1} (C_{t,k} + D_{t,k} \Phi_k) \right\} X_k^{t,\Phi}. \end{aligned}$$

For $k \in \mathbb{T}_t$, let

$$\begin{aligned} \tilde{P}_{t,k} = & Q_{t,k} + \Phi_k^T R_{t,k} \Phi_k + (A_{t,k} + B_{t,k} \Phi_k)^T \tilde{P}_{t,k+1} (A_{t,k} + B_{t,k} \Phi_k) \\ & + (C_{t,k} + D_{t,k} \Phi_k)^T \tilde{P}_{t,k+1} (C_{t,k} + D_{t,k} \Phi_k). \end{aligned}$$

Then, (3.11) holds. Furthermore, by the convexity condition (3.7), we have for any $\bar{u}_t \in L_{\mathcal{F}}^2(t; \mathbb{R}^m)$

$$\begin{aligned} 0 \leq & \mathbb{E}[\bar{u}_t^T R_{t,t} \bar{u}_t] + \mathbb{E}[(Y_N^{t,\bar{u}_t,\Phi})^T \tilde{P}_{t,N} Y_N^{t,\bar{u}_t,\Phi}] + \sum_{\ell=t}^{N-1} \mathbb{E}[(Y_k^{t,\bar{u}_t,\Phi})^T (Q_{t,k} + \Phi_k^T R_{t,k} \Phi_k) Y_k^{t,\bar{u}_t,\Phi}] \\ = & \mathbb{E}[\bar{u}_t^T (R_{t,t} + B_{t,t}^T \tilde{P}_{t,t+1} B_{t,t} + D_{t,t}^T \tilde{P}_{t,t+1} D_{t,t}) \bar{u}_t]. \end{aligned}$$

Hence,

$$\tilde{W}_{t,t} = R_{t,t} + B_{t,t}^T \tilde{P}_{t,t+1} B_{t,t} + D_{t,t}^T \tilde{P}_{t,t+1} D_{t,t} \geq 0.$$

When t ranges from $N-1$ to 0, we have the solvability of (3.9).

(ii) \Rightarrow (i). Note that $\Phi = \{\Phi_0, \dots, \Phi_{N-1}\}$ with $\Phi_t = -\tilde{W}_{t,t}^\dagger \tilde{H}_{t,t}$, $t \in \mathbb{T}$. As (3.9) is solvable, FBS Δ E (3.5) is solvable with property (3.11). Furthermore, by reversing some presentations in “(i) \Rightarrow (ii)”, the stationary condition (3.6) and the convexity condition (3.7) are both satisfied. Therefore, Φ is a linear feedback equilibrium strategy. \square

Substituting Φ into (3.9), one get

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \tilde{P}_{t,k} = Q_{t,k} + A_{t,k}^T \tilde{P}_{t,k+1} A_{t,k} + C_{t,k}^T \tilde{P}_{t,k+1} C_{t,k} - \tilde{H}_{t,k}^T \tilde{W}_{k,k}^\dagger \tilde{H}_{t,k} \\ \quad - \tilde{H}_{t,k}^T \tilde{W}_{k,k}^\dagger \tilde{H}_{k,k} + \tilde{H}_{k,k}^T \tilde{W}_{k,k}^\dagger \tilde{W}_{t,k} \tilde{W}_{k,k}^\dagger \tilde{H}_{k,k} \end{array} \right. \\ \tilde{P}_{t,N} = G_t, \\ k \in \mathbb{T}_t, \\ \tilde{W}_{t,t} \tilde{W}_{t,t}^\dagger \tilde{H}_{t,t} - \tilde{H}_{t,t} = 0, \\ \tilde{W}_{t,t} \geq 0, \\ t \in \mathbb{T}, \end{array} \right. \quad (3.12)$$

where

$$\left\{ \begin{array}{l} \tilde{W}_{t,k} = R_{t,k} + B_{t,k}^T \tilde{P}_{t,k+1} B_{t,k} + D_{t,k}^T \tilde{P}_{t,k+1} D_{t,k}, \\ \tilde{H}_{t,k} = B_{t,k}^T \tilde{P}_{t,k+1} A_{t,k} + D_{t,k}^T \tilde{P}_{t,k+1} C_{t,k}, \\ k \in \mathbb{T}_t, \quad t \in \mathbb{T}. \end{array} \right.$$

For any $t \in \mathbb{T}$, $k \in \mathbb{T}_t$, $\tilde{P}_{t,k}$ is symmetric. Furthermore, if $Q_{t,k} \geq 0$, $R_{t,k} > 0$, $G_t \geq 0$, $k \in \mathbb{T}_t$, $t \in \mathbb{T}$, then we can prove that $\tilde{W}_{t,k} > 0$ and $\tilde{P}_{t,k} \geq 0$, $k \in \mathbb{T}_t$, $t \in \mathbb{T}$. Therefore, (3.12) is solvable.

Corollary 3.4. *If $Q_{t,k} \geq 0, R_{t,k} > 0, G_t \geq 0, k \in \mathbb{T}_t, t \in \mathbb{T}$, then Problem (LQ) admits a unique linear feedback equilibrium strategy Φ .*

For the case where $A_{t,k}, B_{t,k}, C_{t,k}, D_{t,k}, Q_{t,k}, k \in \mathbb{T}_t, G_t$, are all independent of t , (3.12) reads as

$$\left\{ \begin{array}{l} \tilde{P}_{t,k} = Q_k + A_k^T \tilde{P}_{t,k+1} A_k + C_k^T \tilde{P}_{t,k+1} C_k - \tilde{H}_{k,k}^T \tilde{W}_{k,k}^\dagger \tilde{H}_{t,k} \\ \quad - \tilde{H}_{t,k}^T \tilde{W}_{k,k}^\dagger \tilde{H}_{k,k} + \tilde{H}_{k,k}^T \tilde{W}_{k,k}^\dagger \tilde{W}_{t,k} \tilde{W}_{k,k}^\dagger \tilde{H}_{k,k} \\ \tilde{P}_{t,N} = G, \\ k \in \mathbb{T}_t, \\ \tilde{W}_{t,t} \tilde{W}_{t,t}^\dagger \tilde{H}_{t,t} - \tilde{H}_{t,t} = 0, \\ \tilde{W}_{t,t} \geq 0, \\ t \in \mathbb{T}, \end{array} \right. \quad (3.13)$$

where

$$\left\{ \begin{array}{l} \tilde{W}_{t,k} = R_{t,k} + B_k^T \tilde{P}_{t,k+1} B_k + D_k^T \tilde{P}_{t,k+1} D_k, \\ \tilde{H}_{t,k} = B_k^T \tilde{P}_{t,k+1} A_k + D_k^T \tilde{P}_{t,k+1} C_k, \\ k \in \mathbb{T}_t, \quad t \in \mathbb{T}. \end{array} \right.$$

Here, (3.13) is different from (2.41) and (2.42), which relates to the open-loop equilibrium control of the corresponding situation. Moreover, when all the system matrices in the dynamics and cost functional are independent of the initial time, (3.12) will reduce to the standard GDRE [1].

4 Comparison

For the time-inconsistent stochastic LQ problem, open-loop and closed-loop time-consistent solutions are separately investigated in the above two sections. Section 2 is concerned with the open-loop equilibrium control, while the linear feedback equilibrium strategy is studied in Section 3 that is a kind of closed-loop strategy. As noted in Introduction, the objects of study in these two formulations are quite different. Open-loop time-consistent solution is to find an open-loop control that is an equilibrium of a leader-follower game with hierarchical structure. Open-loop control, or simply control, or more exactly control action, in this paper is referred to as a function of time that is also adapted to a filtration; such a meaning of open-loop control is adopted by many scholars in their works (for example, [3] [23]). Generated by some primitive random variables, the filtration is viewed as the “state of nature”, and the adaptedness to such a filtration is to emphasize that the underlining open-loop control is allowed to be random. Note that the key point of above comment is not to the attribute “open-loop” but to the subject “control”.

So far, time-consistent (linear feedback) strategy is the object of closed-loop formulation. A strategy is a decision rule that a controller uses to select his control action based on the available information set. Mathematically, strategy is a mapping or operator on some information set, which is a higher-rank notion other than control. When substituting the available information into a strategy, the open-loop value or open-loop realization of this strategy is then obtained. To get more about this, let us look at the strategy Φ in the definition of linear feedback equilibrium strategy, which can be viewed as a binary mapping $\Phi(\cdot, \cdot)$. When substituting the information $(k, X_k^{t,x,*})$, we have the open-loop value $\Phi(k, X_k^{t,x,*}) = \Phi_k X_k^{t,x,*}$ at time point k . Similarly, $(\Phi X^{k,\Phi})|_{\mathbb{T}_k} = (\Phi(k, X_k^{k,\Phi}), \dots, \Phi(N-1, X_{N-1}^{k,\Phi}))$ is an open-loop value of Φ on the time period \mathbb{T}_k , which is a control action on \mathbb{T}_k .

Besides above conceptual difference between control and strategy, let us further compare the definitions of the two equilibria. In the definition of open-loop equilibrium control, state involved in

$J(k, X_k^{t,x,*}, (u_k, u^{t,x,*}|_{\mathbb{T}_{k+1}}))$ of (2.5) satisfies the following equations

$$\begin{cases} X_{\ell+1}^{k,u_k,u^{t,x,*}} = A_{k,\ell}X_{\ell}^{k,u_k,u^{t,x,*}} + B_{k,\ell}u_{\ell}^{t,x,*} + (C_{k,\ell}X_{\ell}^{k,u_k,u^{t,x,*}} + D_{k,\ell}u_{\ell}^{t,x,*})w_{\ell}, & \ell \in \mathbb{T}_{k+1}, \\ X_{k+1}^{k,u_k,u^{t,x,*}} = A_{k,k}X_k^{k,u_k,u^{t,x,*}} + B_{k,k}u_k^{t,x,*} + (C_{k,k}X_k^{k,u_k,u^{t,x,*}} + D_{k,k}u_k^{t,x,*})w_k, \\ X_k^{k,u_k,u^{t,x,*}} = X_k^{t,x,*}. \end{cases} \quad (4.1)$$

In (4.1), $u^{t,x,*}|_{\mathbb{T}_{k+1}}$ is not influenced by u_k , because it is given prior. On the contrary, in the definition of linear feedback equilibrium strategy, the control $(\Phi X^{k,u_k,\Phi})|_{\mathbb{T}_{k+1}}$ in (3.4) is influenced by u_k via the term $X^{k,u_k,\Phi}$. This makes an essential difference between the open-loop equilibrium control and the linear feedback equilibrium strategy. In addition, an open-loop equilibrium control when being mentioned is corresponding to a fixed initial pair, while the linear feedback equilibrium strategy is required to define for all the initial pairs.

Moreover, the existence of open-loop equilibrium control and linear feedback equilibrium strategy is differently characterized. The fact that there exists an open-loop equilibrium control for any given initial pair, is fully characterized via the solvability of a set of nonsymmetric GDREs and the solvability of a set of LDEs. In contrast, the existence of linear feedback equilibrium strategy is shown to be equivalent to the solvability of a single set of GDREs, which are symmetric. Note that just $R_{k,k}, k \in \mathbb{T}$, are involved in (2.17) and (2.18). Hence, if some $R_{t,k}, k \in \mathbb{T}_{t+1}, t \in \mathbb{T}$, are modified, the existence of open-loop equilibrium control will not change! The reason of this lies in the definition of open-loop equilibrium control. Specifically, given initial pair (t, x) and by subtracting $\sum_{\ell=k+1}^{N-1} \mathbb{E}[(u_{\ell}^{t,x,*})^T R_{k,\ell} u_{\ell}^{t,x,*}]$ from both sides of (2.5), (2.5) can be equivalently rewritten as

$$\begin{aligned} & \mathbb{E} \left[\sum_{\ell=k}^{N-1} (X_{\ell}^{k,u^{t,x,*}})^T Q_{k,\ell} X_{\ell}^{k,u^{t,x,*}} \right] + \mathbb{E} [(u_k^{t,x,*})^T R_{k,k} u_k^{t,x,*}] + \mathbb{E} [(X_N^{k,u^{t,x,*}})^T G_k X_N^{k,u^{t,x,*}}] \\ & \leq \mathbb{E} \left[\sum_{\ell=k}^{N-1} (X_{\ell}^{k,u_k,u^{t,x,*}})^T Q_{k,\ell} X_{\ell}^{k,u_k,u^{t,x,*}} \right] + \mathbb{E} [u_k^T R_{k,k} u_k] + \mathbb{E} [(X_N^{k,u_k,u^{t,x,*}})^T G_k X_N^{k,u_k,u^{t,x,*}}], \end{aligned} \quad (4.2)$$

where $X^{k,u^{t,x,*}}$ and $X^{k,u_k,u^{t,x,*}}$ are given, respectively, by

$$\begin{cases} X_{\ell+1}^{k,u^{t,x,*}} = A_{k,\ell}X_{\ell}^{k,u^{t,x,*}} + B_{k,\ell}u_{\ell}^{t,x,*} + (C_{k,\ell}X_{\ell}^{k,u^{t,x,*}} + D_{k,\ell}u_{\ell}^{t,x,*})w_{\ell}, \\ X_k^{k,u^{t,x,*}} = X_k^{t,x,*}, & \ell \in \mathbb{T}_{k+1}, \end{cases} \quad (4.3)$$

and (4.1). Note that $R_{k,\ell}, \ell \in \mathbb{T}_{k+1}, k \in \mathbb{T}$, do not appear in (4.2). This could explain why only $R_{k,k}, k \in \mathbb{T}$, are involved in (2.17) and (2.18). Otherwise, in the definition of linear feedback equilibrium strategy, $(\Phi X^{k,\Phi})|_{\mathbb{T}_{k+1}}$ and $(\Phi X^{k,u_k,\Phi})|_{\mathbb{T}_{k+1}}$ of (3.1) are different as $X^{k,\Phi}$ differs from $X^{k,u_k,\Phi}$. Therefore, terms in (3.1) associated with $R_{k,\ell}, \ell \in \mathbb{T}_{k+1}$, cannot be removed, which implies the dependence of (3.12) on $R_{k,\ell}, \ell \in \mathbb{T}_{k+1}, k \in \mathbb{T}$.

Furthermore, let us pay attention to the computations of these two equilibria. As noted above, when we mention an open-loop equilibrium control, the initial pair which induces that open-loop equilibrium control should be mentioned simultaneously. Differently, the linear feedback equilibrium strategy is independent of all the initial pairs. These facts will help us to differentiate the backward procedures (mentioned below) of computing an open-loop equilibrium control and the linear feedback equilibrium strategy. Theorem 2.4 and Theorem 3.2 are generally the maximum-principle-type equivalent characterizations, and the backward SΔEs are involved. To obtain an open-loop equilibrium control $u^{t,x,*}$, we should decouple the FBSΔEs (for every k we have a (2.9)) along the equilibrium state $X^{t,x,*}$. Roughly speaking, from the stationary condition, we have $u_{N-1}^{t,x,*}$ by decoupling FBSΔE (2.9) (with $k = N-1$). Generally, based on all the expressions of $u_{\ell}^{t,x,*}, \ell \in \mathbb{T}_{k+1}$, we should recouple the FBSΔE (2.9) on the whole time period \mathbb{T}_k to obtain the linear relation between $Z^{k,*}$ and $X^{k,*}$, and then by the stationary condition we can get $u_k^{t,x,*}$. Note that via its forward initial state $X_k^{t,x,*}$, the FBSΔEs are

coupled with the equilibrium state $X^{t,x,*}$, and the equilibrium control and the equilibrium state are obtained simultaneously.

Concerned with the linear feedback equilibrium strategy, if exists, it can be calculated via solving the GDREs (3.12) backwardly. For every generic time point t , we need to solve a single GDRE (associated with t) over the whole time period \mathbb{T}_t , and use the solution $\tilde{P}_{t,t+1}$ at time point $t+1$ to construct $\Phi_t = \tilde{W}_{t,t}^\dagger \tilde{H}_{t,t}$. This backward procedure is also due to a decoupling procedure of FBSΔEs (3.5). It should be mention that only under the condition of the existence of linear feedback equilibrium strategy, this strategy can be computed backwardly as above. Furthermore, necessary and sufficient conditions on the existence of the equilibria are the main concerns of this paper, from which the computations of the equilibria are much direct.

To end this section, we give the following final comment. Under the condition that there exists an open-loop equilibrium control for any given initial pair, all the open-loop equilibrium controls happen to be of feedback form; while the feedback gain $\{-W_{k,k}^\dagger H_{k,k}, k \in \mathbb{T}\}$ in (2.20) is not a linear feedback equilibrium strategy of Problem (LQ) indeed!

5 Examples

In this section, we shall present three examples to illustrate the theory derived above.

Example 5.1. *Let*

$$\begin{aligned} A_0 &= \begin{bmatrix} 1.12 & 0.21 \\ -0.13 & 0.98 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2.12 & -0.35 \\ -0.21 & 3.43 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 5.46 & 1.21 \\ -0.98 & 4.21 \end{bmatrix}, \\ B_0 &= \begin{bmatrix} 1.45 & -0.23 \\ -0.2 & 4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1.5 & 0.3 \\ -0.2 & 3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -4.36 & 0.82 \\ 1.21 & 4.21 \end{bmatrix}, \\ C_0 &= \begin{bmatrix} 1 & 0.32 \\ 0.25 & 3 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1.65 & -0.13 \\ -0.42 & 6 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -3 & 1.53 \\ -0.62 & 4.78 \end{bmatrix}, \\ D_0 &= \begin{bmatrix} 5 & 1 \\ -0.85 & 8 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 4 & 0.53 \\ -0.42 & 5 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 9.21 & -2.03 \\ -1.52 & 6.98 \end{bmatrix}, \\ Q_0 &= \begin{bmatrix} -2 & 0.8 \\ 0.8 & 1.6 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1.56 & -0.23 \\ -0.23 & 2.54 \end{bmatrix}, \quad R_{0,0} = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix}, \\ R_{0,1} &= \begin{bmatrix} -5 & 0 \\ 0 & -4 \end{bmatrix}, \quad R_{0,2} = \begin{bmatrix} -9 & 0 \\ 0 & 10 \end{bmatrix}, \quad R_{1,1} = \begin{bmatrix} 4 & -0.3 \\ -0.3 & 7 \end{bmatrix}, \\ R_{1,2} &= \begin{bmatrix} 2.24 & -5.67 \\ -5.67 & -1.27 \end{bmatrix}, \quad R_{2,2} = \begin{bmatrix} 6.29 & -1.67 \\ -1.67 & 8.38 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

Here, $Q_1 \geq 0, Q_2, G, R_{1,1}, R_{2,2} > 0$; $Q_0, R_{0,0}$ and $R_{1,2}$ are indefinite, and $R_{0,1}$ is negative definite.

By (2.41) and (2.42), we get

$$\begin{aligned} P_2 &= \begin{bmatrix} 16.6571 & 5.8520 \\ 5.8520 & 11.5436 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 6.9700 & -1.3882 \\ -1.3882 & 9.1396 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 7.8991 & 4.2276 \\ 4.2276 & 4.6336 \end{bmatrix}, \\ S_2 &= \begin{bmatrix} 43.0612 & -12.3922 \\ -12.3922 & 87.4900 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 11.6579 & -7.4371 \\ -7.4371 & 95.6692 \end{bmatrix}, \quad S_0 = \begin{bmatrix} 34.3248 & 35.9699 \\ 35.9699 & 938.8710 \end{bmatrix} \end{aligned}$$

with

$$W_2 = \begin{bmatrix} 117.6727 & -34.9725 \\ -34.9725 & 146.0623 \end{bmatrix} > 0, \quad W_1 = \begin{bmatrix} 287.3160 & 153.0623 \\ 153.0623 & 447.2115 \end{bmatrix} > 0, \quad (5.1)$$

$$W_0 = \begin{bmatrix} 1138.3 & -410.9 \\ -410.9 & 915.8 \end{bmatrix} > 0, \quad (5.2)$$

$$R_{2,2} + B_2^T S_3 B_2 + D_2^T S_3 D_2 = \begin{bmatrix} 117.6727 & -34.9725 \\ -34.9725 & 146.0623 \end{bmatrix} > 0, \quad (5.3)$$

$$R_{1,1} + B_1^T S_2 B_1 + D_1^T S_2 D_1 = \begin{bmatrix} 857.9 & -426.0 \\ -426.0 & 2909.6 \end{bmatrix} > 0, \quad (5.4)$$

$$R_{0,0} + B_0^T S_1 B_0 + D_0^T S_1 D_0 = \begin{bmatrix} 17940 & -54740 \\ -54740 & 331470 \end{bmatrix} > 0. \quad (5.5)$$

From (5.1)-(5.5), the corresponding (2.41) and (2.42) are solvable, and thus, the open-loop equilibrium pair exists. Furthermore, an open-loop equilibrium control for the initial pair $(0, x)$ is given by

$$u_k^{0,x,*} = -W_k^\dagger H_k X_k^{0,x,*}, \quad k = 0, 1, 2$$

with

$$\begin{aligned} -W_0^\dagger H_0 &= \begin{bmatrix} -0.2183 & 0.0031 \\ 0.0023 & -0.3286 \end{bmatrix}, \quad -W_1^\dagger H_1 = \begin{bmatrix} -0.5138 & 0.1973 \\ 0.0026 & -1.1339 \end{bmatrix}, \\ -W_2^\dagger H_2 &= \begin{bmatrix} 0.4889 & -0.2601 \\ 0.1605 & -0.7474 \end{bmatrix} \end{aligned}$$

and

$$\begin{cases} X_{k+1}^{0,x,*} = (A_k X_k^{0,x,*} + B_k u_k^{0,x,*}) \\ \quad + (C_k X_k^{0,x,*} + D_k u_k^{0,x,*}) w_k, \\ X_0^{0,x,*} = x, \quad k = 0, 1, 2. \end{cases}$$

On the other hand, from (3.13) we can obtain the solution. However, we have

$$\widetilde{W}_{1,1} = \begin{bmatrix} 239.0218 & 247.7565 \\ 247.7565 & 224.5117 \end{bmatrix},$$

whose eigenvalues are

$$\lambda_1 = -16.096, \quad \lambda_2 = 479.6294.$$

Clearly, $\widetilde{W}_{1,1}$ is indefinite, and thus, the corresponding (3.13) is not solvable. This means that the linear feedback equilibrium strategy does not exist.

Example 5.2. *The system matrices and the weight matrices are the same as those of Example 5.1 except for $R_{0,1}, R_{0,2}, R_{1,2}$, which are now*

$$R_{0,1} = \begin{bmatrix} -1 & 0 \\ 0 & -0.6 \end{bmatrix}, \quad R_{0,2} = \begin{bmatrix} 9.45 & 1.32 \\ 1.32 & 10.78 \end{bmatrix}, \quad R_{1,2} = \begin{bmatrix} 5.24 & -1.67 \\ -1.67 & 7.27 \end{bmatrix}.$$

Note that $R_{0,1}$ is negative definite.

In this case, the corresponding (2.41) and (2.42) are the same as those of Example 5.1, because $R_{0,1}, R_{0,2}, R_{1,2}$ do not enter the GDRE and LDE. Hence, the open-loop equilibrium pair exists for any initial pair. Let us check the existence of the linear feedback equilibrium strategy. By (3.13) we have

$$\begin{aligned} \widetilde{P}_{2,2} &= \begin{bmatrix} 16.6571 & 5.8520 \\ 5.8520 & 11.5436 \end{bmatrix}, \quad \widetilde{P}_{1,2} = \begin{bmatrix} 16.3775 & 6.1187 \\ 6.1187 & 10.8526 \end{bmatrix}, \quad \widetilde{P}_{1,1} = \begin{bmatrix} 37.3769 & -10.7301 \\ -10.7301 & 12.7823 \end{bmatrix}, \\ \widetilde{P}_{0,2} &= \begin{bmatrix} 17.9435 & 3.9449 \\ 3.9449 & 14.2605 \end{bmatrix}, \quad \widetilde{P}_{0,1} = \begin{bmatrix} 39.7057 & -11.0096 \\ -11.0096 & 3.2054 \end{bmatrix}, \quad \widetilde{P}_{0,0} = \begin{bmatrix} 6.1615 & 4.3853 \\ 4.3853 & 3.2889 \end{bmatrix} \end{aligned}$$

with

$$\widetilde{W}_{2,2} = \begin{bmatrix} 117.6727 & -34.9725 \\ -34.9725 & 146.0623 \end{bmatrix} > 0, \quad \widetilde{W}_{1,1} = \begin{bmatrix} 281.0078 & 160.6675 \\ 160.6675 & 425.5062 \end{bmatrix} > 0, \quad (5.6)$$

$$\widetilde{W}_{0,0} = \begin{bmatrix} 1178.0 & -334.5 \\ -334.5 & 143.3 \end{bmatrix} > 0. \quad (5.7)$$

It follows from (5.6)-(5.7) that (3.13) is solvable. Furthermore, a linear feedback equilibrium strategy (Φ_0, Φ_1, Φ_2) is given by

$$\Phi_0 = \begin{bmatrix} -0.0368 & 0.0884 \\ 0.6555 & -0.0192 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} -0.5094 & 0.1935 \\ -0.0021 & -1.1301 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 0.4889 & -0.2601 \\ 0.1605 & -0.7474 \end{bmatrix}.$$

Example 5.3. *Let*

$$\begin{aligned} A_{0,0} &= \begin{bmatrix} 2.3 & 0.41 \\ -0.3 & 1.9 \end{bmatrix}, \quad A_{0,1} = \begin{bmatrix} 4.12 & -0.35 \\ 0.31 & 3.03 \end{bmatrix}, \quad B_{0,0} = \begin{bmatrix} 2.45 & -0.3 \\ 0.2 & 4 \end{bmatrix}, \\ B_{0,1} &= \begin{bmatrix} 2.5 & 0.6 \\ -0.2 & 3 \end{bmatrix}, \quad C_{0,0} = \begin{bmatrix} 2.2 & 0.32 \\ 0.5 & 3 \end{bmatrix}, \quad C_{0,1} = \begin{bmatrix} 3.65 & -0.3 \\ -0.42 & 5.6 \end{bmatrix}, \\ D_{0,0} &= \begin{bmatrix} 5.6 & 1 \\ 0.73 & 7.8 \end{bmatrix}, \quad D_{0,1} = \begin{bmatrix} 5 & 0.73 \\ -0.47 & 5.2 \end{bmatrix}, \quad A_{1,1} = \begin{bmatrix} 6 & 1.63 \\ -1.37 & 7 \end{bmatrix}, \\ B_{1,1} &= \begin{bmatrix} 4 & 0.93 \\ 1.07 & 3 \end{bmatrix}, \quad C_{1,1} = \begin{bmatrix} 8 & 2.03 \\ -1.23 & 10 \end{bmatrix}, \quad D_{1,1} = \begin{bmatrix} 5 & -0.93 \\ 1.016 & 4.65 \end{bmatrix}, \\ Q_{0,0} &= \begin{bmatrix} 2 & 0.8 \\ 0.8 & 1.6 \end{bmatrix}, \quad Q_{0,1} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_{0,0} = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_{0,1} = \begin{bmatrix} -5 & 0 \\ 0 & -4 \end{bmatrix}, \\ Q_{1,1} &= \begin{bmatrix} 2 & 0.1 \\ 0.1 & 5 \end{bmatrix}, \quad R_{1,1} = \begin{bmatrix} 4 & -0.3 \\ -0.3 & 7 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 2 & -0.3 \\ -0.3 & 3 \end{bmatrix}. \end{aligned}$$

Note that $R_{0,0}$ is indefinite and $R_{0,1}$ is negative definite.

By (3.12), we have

$$\widetilde{P}_{1,1} = \begin{bmatrix} 18.8304 & -11.9513 \\ -11.9513 & 46.5418 \end{bmatrix}, \quad \widetilde{P}_{0,1} = \begin{bmatrix} 40.6027 & -28.7266 \\ -28.7266 & 50.9647 \end{bmatrix}, \quad \widetilde{P}_{0,0} = \begin{bmatrix} 99.6787 & 14.1112 \\ 14.1112 & 8.3265 \end{bmatrix}$$

with

$$\widetilde{W}_{1,1} = \begin{bmatrix} 86.9155 & 11.0531 \\ 11.0531 & 103.2478 \end{bmatrix} > 0, \quad \widetilde{W}_{0,0} = \begin{bmatrix} 1282.7 & -1027.0 \\ -1027.0 & 3582.2 \end{bmatrix} > 0. \quad (5.8)$$

It follows from (5.8) that the corresponding (3.12) is solvable. Hence, the linear feedback equilibrium strategy does exist. Furthermore, a linear feedback equilibrium strategy (Φ_0, Φ_1) is given by

$$\Phi_0 = \begin{bmatrix} -0.4665 & -0.0206 \\ 0.0269 & -0.3965 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} -1.4499 & -0.4726 \\ 0.6369 & -1.8700 \end{bmatrix}.$$

6 Conclusion

In this paper, we investigated the open-loop equilibrium control and the linear feedback equilibrium strategy of the time-inconsistent indefinite stochastic LQ optimal control. Necessary and sufficient conditions are presented for these two cases, respectively. Furthermore, the GDREs and LDEs are introduced to characterize the linear feedback form of the open-loop equilibrium control and the linear feedback equilibrium strategy. For future researches, we would like to study the time-inconsistent problems for jump parameter systems [6], Boolean networks [12], multi-agent systems [20] etc., and extend the methodology developed in this paper to other types of time-inconsistency.

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Appendix

A. Proof of Theorem 2.4

Denote $J(k, X_k^{t,x,*}; (u_k, u^{t,x,*}|_{\mathbb{T}_{k+1}}))$ by $\bar{J}(k, X_k^{t,x,*}; u_k)$. Then, by (2.5), we have

$$\bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*}) \leq \bar{J}(k, X_k^{t,x,*}; u_k). \quad (6.1)$$

This means that $u_k^{t,x,*}$ is an optimal control of the following nonstandard optimal control problem (denoted as Problem (LQ)_k):

$$\left\{ \begin{array}{l} \text{Minimize } \bar{J}(k, X_k^{t,x,*}; u_k) \text{ over } L^2_{\mathcal{F}}(k; \mathbb{R}^m), \\ \text{subject to} \\ \left\{ \begin{array}{l} X_{\ell+1}^k = A_{k,\ell} X_{\ell}^k + B_{k,\ell} u_{\ell}^{t,x,*} + (C_{k,\ell} X_{\ell}^k + D_{k,\ell} u_{\ell}^{t,x,*}) w_{\ell}, \\ X_{k+1}^k = (A_{k,k} X_k^k + B_{k,k} u_k) + (C_{k,k} X_k^k + D_{k,k} u_k) w_k, \\ X_k^k = X_k^{t,x,*}, \quad \ell \in \mathbb{T}_{k+1}. \end{array} \right. \end{array} \right.$$

Here, we call Problem (LQ) $_k$ a non-standard optimal control problem as $u^{t,x,*}|_{\mathbb{T}_{k+1}}$ in the dynamics of X^k is fixed, and we just select u_k to minimize $\bar{J}(k, X_k^{t,x,*}; u_k)$.

To proceed, we introduce an inner product on $L_{\mathcal{F}}^2(\mathbb{T}_k; \mathbb{R}^p)$ with $p = n, m$, and $k \in \tilde{\mathbb{T}}_t$:

$$\langle y, z \rangle_{\mathbb{T}_k} = \sum_{\ell=k}^{N-1} \mathbb{E}(y_{\ell}^T z_{\ell}), \text{ for } y, z \in L_{\mathcal{F}}^2(\mathbb{T}_k; \mathbb{R}^p),$$

and use the convention

$$\begin{cases} (Q_k x)(\cdot) = Q_{k,\cdot} x, & \forall x \in L_{\mathcal{F}}^2(\mathbb{T}_k; \mathbb{R}^n), \\ (R_k + u)(\cdot) = R_{k,\cdot} u, & \forall u \in L_{\mathcal{F}}^2(\mathbb{T}_{k+1}; \mathbb{R}^m). \end{cases} \quad (6.2)$$

For $L_{\mathcal{F}}^2(k; \mathbb{R}^p)$ with $p = n, m$, and $k \in \tilde{\mathbb{T}}_t$, the inner product is defined as

$$\langle y, z \rangle_k = \mathbb{E}(y^T z), \text{ for } y, z \in L_{\mathcal{F}}^2(k; \mathbb{R}^p).$$

Then, the cost functional $\bar{J}(k, X_k^{t,x,*}; u_k)$ can be rewritten as

$$\begin{aligned} \bar{J}(k, X_k^{t,x,*}; u_k) &= \langle Q_k X^k, X^k \rangle_{\mathbb{T}_k} + \langle R_{k,k} u_k, u_k \rangle_k + \langle R_{k+} u^{t,x,*}|_{\mathbb{T}_{k+1}}, u^{t,x,*}|_{\mathbb{T}_{k+1}} \rangle_{\mathbb{T}_{k+1}} \\ &\quad + \langle G_k X_N^k, X_N^k \rangle_N. \end{aligned} \quad (6.3)$$

We now calculate the first order and second order directional derivatives of $\bar{J}(k, X_k^{t,x,*}; u_k)$ at $u_k^{t,x,*}$. Corresponding to controls $u^{t,x,*}|_{\mathbb{T}_k}$ and $(u_k^{t,x,*} + \lambda \bar{u}_k, u^{t,x,*}|_{\mathbb{T}_{k+1}})$, the solutions of (2.8) with the initial state $X_k^{t,x,*}$ are, respectively, denoted by $X^{k,*}$ and $X^{k,\lambda}$. Then, we have

$$\begin{cases} \frac{X_{\ell+1}^{k,\lambda} - X_{\ell+1}^{k,*}}{\lambda} = A_{k,\ell} \frac{X_{\ell}^{k,\lambda} - X_{\ell}^{k,*}}{\lambda} + C_{k,\ell} \frac{X_{\ell}^{k,\lambda} - X_{\ell}^{k,*}}{\lambda} w_{\ell}, \\ \frac{X_{k+1}^{k,\lambda} - X_{k+1}^{k,*}}{\lambda} = \left(A_{k,k} \frac{X_k^{k,\lambda} - X_k^{k,*}}{\lambda} + B_{k,k} \bar{u}_k \right) + \left(C_{k,k} \frac{X_k^{k,\lambda} - X_k^{k,*}}{\lambda} + D_{k,k} \bar{u}_k \right) w_k, \\ \frac{X_{\ell}^{k,\lambda} - X_{\ell}^{k,*}}{\lambda} = 0, \quad \ell \in \mathbb{T}_{k+1}, \end{cases}$$

which can be rewritten as

$$\begin{cases} Y_{\ell+1}^k = A_{k,\ell} Y_{\ell}^k + C_{k,\ell} Y_{\ell}^k w_{\ell}, & \ell \in \mathbb{T}_{k+1}, \\ Y_{k+1}^k = B_{k,k} \bar{u}_k + D_{k,k} \bar{u}_k w_k, \\ Y_k^k = 0 \end{cases} \quad (6.4)$$

with $\frac{X_{\ell}^{k,\lambda} - X_{\ell}^{k,*}}{\lambda} = Y_{\ell}^k$. For any $\ell \in \mathbb{T}_k$, we have $X_{\ell}^{k,\lambda} = X_{\ell}^{k,*} + \lambda Y_{\ell}^k$. To proceed, some calculations show

$$\begin{aligned} &\lim_{\lambda \downarrow 0} \frac{\langle R_{k,k}(u_k^{t,x,*} + \lambda \bar{u}_k), u_k^{t,x,*} + \lambda \bar{u}_k \rangle_k - \langle R_{k,k} u_k^{t,x,*}, u_k^{t,x,*} \rangle_k}{\lambda} \\ &= 2 \langle R_{k,k} u_k^{t,x,*}, \bar{u}_k \rangle_k + \lim_{\lambda \downarrow 0} \lambda \langle R_{k,k} \bar{u}_k, \bar{u}_k \rangle_k \\ &= 2 \langle R_{k,k} u_k^{t,x,*}, \bar{u}_k \rangle_k, \end{aligned}$$

and

$$\begin{aligned} &\lim_{\lambda \downarrow 0} \frac{\langle Q_k X^{k,\lambda}, X^{k,\lambda} \rangle_{\mathbb{T}_k} - \langle Q_k X^{k,*}, X^{k,*} \rangle_{\mathbb{T}_k}}{\lambda} = 2 \langle Q_k X^{k,*}, Y^k \rangle_{\mathbb{T}_k}, \\ &\lim_{\lambda \downarrow 0} \frac{\langle G_k X_N^{k,\lambda}, X_N^{k,\lambda} \rangle_N - \langle G_k X_N^{k,*}, X_N^{k,*} \rangle_N}{\lambda} = 2 \langle G_k X_N^{k,*}, Y_N^k \rangle_N. \end{aligned}$$

Hence, we have the first order directional derivative of $\bar{J}(k, X_k^{t,x,*}; u_k)$ at $u_k^{t,x,*}$ with the direction \bar{u}_k

$$d\bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*}; \bar{u}_k) = \lim_{\lambda \downarrow 0} \frac{\bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*} + \lambda \bar{u}_k) - \bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*})}{\lambda}$$

$$= 2\langle Q_k X^{k,*}, Y^k \rangle_{\mathbb{T}_k} + 2\langle R_{k,k} u_k^{t,x,*}, \bar{u}_k \rangle_k + 2\langle G_k X_N^{k,*}, Y_N^k \rangle_N. \quad (6.5)$$

Similarly, the second order directional derivative with the direction (\bar{u}_k, \hat{u}_k) is given by

$$\begin{aligned} d^2 \bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*}; \bar{u}_k; \hat{u}_k) &= \lim_{\beta \downarrow 0} \frac{d\bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*} + \beta \hat{u}_k; \bar{u}_k) - d\bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*}; \bar{u}_k)}{\beta} \\ &= 2\langle Q_k \hat{Y}^k, Y^k \rangle_{\mathbb{T}_k} + 2\langle R_{k,k} \hat{u}_k, \bar{u}_k \rangle_k + 2\langle G_k \hat{Y}_N^k, Y_N^k \rangle_N, \end{aligned}$$

where

$$\begin{cases} \hat{Y}_{\ell+1}^k = A_{k,\ell} \hat{Y}_\ell^k + C_{k,\ell} \hat{Y}_\ell^k w_\ell, & \ell \in \mathbb{T}_{k+1}, \\ \hat{Y}_{k+1}^k = B_{k,k} \bar{u}_k + D_{k,k} \bar{u}_k w_k, \\ \hat{Y}_k^k = 0. \end{cases}$$

If $\hat{u}_k = \bar{u}_k$, then

$$d^2 \bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*}; \bar{u}_k; \bar{u}_k) = 2\langle Q_k Y^k, Y^k \rangle_{\mathbb{T}_k} + 2\langle R_{k,k} \bar{u}_k, \bar{u}_k \rangle_k + 2\langle G_k Y_N^k, Y_N^k \rangle_N. \quad (6.6)$$

Note that the righthand side of (6.6) is independent of $u_k^{t,x,*}$. Then, for any $u_k \in L_{\mathcal{F}}^2(k; \mathbb{R}^m)$, we have

$$d^2 \bar{J}(k, X_k^{t,x,*}; u_k; \bar{u}_k; \bar{u}_k) = 2\langle Q_k Y^k, Y^k \rangle_{\mathbb{T}_k} + 2\langle R_{k,k} \bar{u}_k, \bar{u}_k \rangle_k + 2\langle G_k Y_N^k, Y_N^k \rangle_N. \quad (6.7)$$

Furthermore, we can show that $\bar{J}(k, X_k^{t,x,*}; u_k)$ is infinitely differentiable with respect to u_k in the sense that the directional derivatives of all orders exist. By classical results on convex analysis [8], we have the following result.

Lemma 6.1. *The following statements are equivalent.*

- (i) *The map $u_k \mapsto \bar{J}(k, X_k^{t,x,*}; u_k)$ is convex.*
- (ii) *The following holds*

$$\inf_{\bar{u}_k \in L_{\mathcal{F}}^2(k; \mathbb{R}^m)} [\langle Q_k Y^k, Y^k \rangle_{\mathbb{T}_k} + \langle R_{k,k} \bar{u}_k, \bar{u}_k \rangle_k + \langle G_k Y_N^k, Y_N^k \rangle_N] \geq 0.$$

Proof of Theorem 2.4. (i) \Rightarrow (ii). Let $u^{t,x,*}$ be an open-loop equilibrium control of Problem (LQ) for the initial pair (t, x) . Since $\bar{J}(k, X_k^{t,x,*}; u_k)$ is infinitely differentiable with respect to u_k and (6.7) is independent of u_k , the minimum point $u_k^{t,x,*}$ of $\bar{J}(k, X_k^{t,x,*}; u_k)$ is fully characterized via the first and second order derivatives, namely, $d\bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*}; \bar{u}_k) = 0$ and $d^2 \bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*}; \bar{u}_k; \bar{u}_k) \geq 0$ hold for any \bar{u}_k in $L_{\mathcal{F}}^2(k; \mathbb{R}^m)$. Following Lemma 6.1, (2.11) holds. The forward SΔE of $X^{k,*}$ is clearly solvable as $Z^{k,*}$ does not appear in this SΔE. After obtaining $X^{k,*}$ and substituting $X^{k,*}$ into the backward SΔE, we then have $Z^{k,*}$. This means that the FBSΔE (2.9) admits a solution $(X^{k,*}, Z^{k,*})$. Furthermore, by (6.5), one can get

$$\begin{aligned} \frac{1}{2} d^2 \bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*}; \bar{u}_k) &= \sum_{\ell=k}^{N-1} \mathbb{E}[(X_\ell^{k,*})^T Q_{k,\ell} Y_\ell^k] + \mathbb{E}[(u_k^{t,x,*})^T R_{k,k} \bar{u}_k] + \mathbb{E}[(X_N^{k,*})^T G_k Y_N^k] \\ &= \sum_{\ell=k}^{N-1} \mathbb{E}[(X_\ell^{k,*})^T Q_{k,\ell} Y_\ell^k] + \mathbb{E}[(u_k^{t,x,*})^T R_{k,k} \bar{u}_k] + \sum_{\ell=k}^{N-1} \mathbb{E}[(Z_{\ell+1}^{k,*})^T Y_{\ell+1}^k - (Z_\ell^{k,*})^T Y_\ell^k] \\ &= \sum_{\ell=k}^{N-1} \mathbb{E} \left\{ \left[A_{k,\ell}^T \mathbb{E}(Z_{\ell+1}^{k,*} | \mathcal{F}_{\ell-1}) + Q_{k,\ell} X_\ell^{k,*} + C_{k,\ell}^T \mathbb{E}(Z_{\ell+1}^{k,*} w_\ell | \mathcal{F}_{\ell-1}) - Z_\ell^{k,*} \right]^T Y_\ell^k \right\} \\ &\quad + \mathbb{E} \left\{ \left[R_{k,k} u_k^{t,x,*} + B_{k,k}^T \mathbb{E}(Z_{k+1}^{k,*} | \mathcal{F}_{k-1}) + D_{k,k}^T \mathbb{E}(Z_{k+1}^{k,*} w_t | \mathcal{F}_{k-1}) \right]^T \bar{u}_k \right\}. \end{aligned} \quad (6.8)$$

As $d\bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*}; \bar{u}_k) = 0$ for all $\bar{u}_k \in L_{\mathcal{F}}^2(k; \mathbb{R}^m)$ and the FBSΔE (2.9) is solvable, the stationary condition (2.10) follows.

(ii) \Rightarrow (i). Note that (6.8), (2.9) and (2.10), we have

$$d\bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*}; \bar{u}_k) = 0, \quad \forall \bar{u}_k \in L_{\mathcal{F}}^2(k; \mathbb{R}^m),$$

which together with (2.11) implies that $u_k^{t,x,*}$ is a minimizer of $\bar{J}(k, X_k^{t,x,*}; u_k)$ over $L_{\mathcal{F}}^2(k; \mathbb{R}^m)$. Thus, (6.1), equivalently (2.5), holds for $k \in \mathbb{T}_t$. This proves the conclusion. \square

B. Proof of Theorem 3.2

As (3.1) holds for any $u_k \in L_{\mathcal{F}}^2(k; \mathbb{R}^m)$, in Definition 3.1 (3.1) can be equivalently replaced by

$$J(k, X_k^{t,x,*}; (\Phi X^{k,\Phi})|_{\mathbb{T}_k}) \leq J(k, X_k^{t,x,*}; (u_k + \Phi_k \bar{X}_k^{k,u_k,\Phi}, (\Phi \bar{X}_k^{k,u_k,\Phi})|_{\mathbb{T}_{k+1}})), \quad (6.9)$$

where

$$\begin{cases} \bar{X}_{\ell+1}^{k,u_k,\Phi} = (A_{k,\ell} + B_{k,\ell}\Phi_\ell) \bar{X}_\ell^{k,u_k,\Phi} + (C_{k,\ell} + D_{k,\ell}\Phi_\ell) \bar{X}_\ell^{k,u_k,\Phi} w_\ell, & \ell \in \mathbb{T}_{k+1}, \\ \bar{X}_{k+1}^{k,u_k,\Phi} = (A_{k,k} + B_{k,k}\Phi_k) \bar{X}_k^{k,u_k,\Phi} + B_{k,k}u_k + [(C_{k,k} + D_{k,k}\Phi_k) \bar{X}_k^{k,u_k,\Phi} + D_{k,k}u_k] w_k, \\ \bar{X}_k^{k,u_k,\Phi} = X_k^{t,x,*}. \end{cases}$$

For any $u \in L_{\mathcal{F}}^2(\mathbb{T}; \mathbb{R}^m)$, $k \in \mathbb{T}$ and $y \in L_{\mathcal{F}}^2(k; \mathbb{R}^m)$, let

$$\hat{J}(k, y; u|_{\mathbb{T}_k}) = \mathbb{E}[(X_N^k)^T G_k X_N^k] + \sum_{\ell=k}^{N-1} \mathbb{E}[(X_\ell^k)^T Q_{k,\ell} X_\ell^k + (\Phi_\ell X_\ell^k + u_\ell)^T R_{k,\ell} (\Phi_\ell X_\ell^k + u_\ell)], \quad (6.10)$$

where

$$\begin{cases} X_{\ell+1}^k = (A_{k,\ell} + B_{k,\ell}\Phi_\ell) X_\ell^k + B_{k,\ell}u_\ell + [(C_{k,\ell} + D_{k,\ell}\Phi_\ell) X_\ell^k + D_{k,\ell}u_\ell] w_\ell, \\ X_k^k = y, \quad \ell \in \mathbb{T}_k. \end{cases}$$

Hence, (6.9) reads as

$$\hat{J}(k, X_k^{t,x,*}; 0|_{\mathbb{T}_k}) \leq \hat{J}(k, X_k^{t,x,*}; (u_k, 0|_{\mathbb{T}_{k+1}})), \quad (6.11)$$

where $0|_{\mathbb{T}_k}$ is understood as $u|_{\mathbb{T}_k} = \{u_k, \dots, u_{N-1}\}$ with $u_\ell = 0, \ell \in \mathbb{T}_k$, and similar meaning holds for $0|_{\mathbb{T}_{k+1}}$. Therefore, if Φ is a linear feedback equilibrium strategy of Problem (LQ), then $0|_{\mathbb{T}}$ will be an open-loop equilibrium control of the time-inconsistent stochastic LQ problem corresponding to (2.1) and (6.10). The cost functional (6.10) is different from (2.3), as in (6.10) crossing terms between X_ℓ^k and u_ℓ appear. The following is to mimic the proof of Theorem 2.4.

For any given $\Phi = \{\Phi_0, \dots, \Phi_{N-1}\}$ with $\Phi_t \in \mathbb{R}^{m \times n}, t \in \mathbb{T}$, denote above $\hat{J}(k, X_k^{t,x,*}; (u_k, 0|_{\mathbb{T}_{k+1}}))$ and $\hat{J}(k, X_k^{t,x,*}; 0|_{\mathbb{T}_k})$ by $\tilde{J}(k, X_k^{t,x,*}; u_k)$ and $\tilde{J}(k, X_k^{t,x,*}; 0)$, respectively. We now calculate the first two orders directional derivatives of $\tilde{J}(k, X_k^{t,x,*}; u_k)$. Note that

$$\begin{aligned} \tilde{J}(k, X_k^{t,x,*}; u_k) &= \langle Q_k \bar{X}_k^{k,u_k,\Phi}, \bar{X}_k^{k,u_k,\Phi} \rangle_{\mathbb{T}_k} + \langle R_{k,k}(u_k + \Phi_k \bar{X}_k^{k,u_k,\Phi}), u_k + \Phi_k \bar{X}_k^{k,u_k,\Phi} \rangle_k \\ &\quad + \langle R_{k+}(\Phi \bar{X}_k^{k,u_k,\Phi})|_{\mathbb{T}_{k+1}}, (\Phi \bar{X}_k^{k,u_k,\Phi})|_{\mathbb{T}_k} \rangle_{\mathbb{T}_k} + \langle G_k \bar{X}_N^{k,u_k,\Phi}, \bar{X}_N^{k,u_k,\Phi} \rangle_N, \end{aligned}$$

and

$$\begin{aligned} \tilde{J}(k, X_k^{t,x,*}; u_k + \lambda \bar{u}_k) &= \langle Q_k \bar{X}_k^{k,u_k,\Phi,\lambda}, \bar{X}_k^{k,u_k,\bar{u}_k,\Phi,\lambda} \rangle_{\mathbb{T}_k} \\ &\quad + \langle R_{k+}(\Phi \bar{X}_k^{k,u_k,\bar{u}_k,\Phi,\lambda})|_{\mathbb{T}_{k+1}}, (\Phi \bar{X}_k^{k,u_k,\bar{u}_k,\Phi,\lambda})|_{\mathbb{T}_{k+1}} \rangle_{\mathbb{T}_{k+1}} \\ &\quad + \langle R_{k,k}(u_k + \Phi_k \bar{X}_k^{k,u_k,\bar{u}_k,\Phi,\lambda} + \lambda \bar{u}_k), u_k + \Phi_k \bar{X}_k^{k,u_k,\bar{u}_k,\Phi,\lambda} \\ &\quad + \lambda \bar{u}_k \rangle_k + \langle G_k \bar{X}_N^{k,u_k,\bar{u}_k,\Phi,\lambda}, \bar{X}_N^{k,u_k,\bar{u}_k,\Phi,\lambda} \rangle_N, \end{aligned}$$

where Q_k, R_{k+} are similarly defined as those in (6.2), and

$$\begin{cases} \overline{X}_{\ell+1}^{k, u_k, \bar{u}_k, \Phi, \lambda} = (A_{k, \ell} + B_{k, \ell} \Phi_\ell) \overline{X}_\ell^{k, u_k, \bar{u}_k, \Phi, \lambda} + (C_{k, \ell} + D_{k, \ell} \Phi_\ell) \overline{X}_\ell^{k, u_k, \bar{u}_k, \Phi, \lambda} w_\ell, & \ell \in \mathbb{T}_{k+1}, \\ \overline{X}_{k+1}^{k, u_k, \bar{u}_k, \Phi, \lambda} = (A_{k, k} + B_{k, k} \Phi_k) \overline{X}_k^{k, u_k, \bar{u}_k, \Phi, \lambda} + B_{k, k} (u_k + \lambda \bar{u}_k) \\ \quad + [(C_{k, k} + D_{k, k} \Phi_k) \overline{X}_k^{k, u_k, \bar{u}_k, \Phi, \lambda} + D_{k, k} (u_k + \lambda \bar{u}_k)] w_k, \\ \overline{X}_k^{k, u_k, \bar{u}_k, \Phi, \lambda} = X_k^{t, x, *}. \end{cases}$$

As $\overline{X}_k^{k, u_k, \Phi} = \overline{X}_k^{k, u_k, \bar{u}_k, \Phi, \lambda} = X_k^{t, x, *}$, we have

$$\begin{cases} \frac{\overline{X}_{\ell+1}^{k, u_k, \bar{u}_k, \Phi, \lambda} - \overline{X}_{\ell+1}^{k, u_k, \Phi}}{\lambda} = (A_{k, \ell} + B_{k, \ell} \Phi_\ell) \frac{\overline{X}_\ell^{k, u_k, \bar{u}_k, \Phi, \lambda} - \overline{X}_\ell^{k, u_k, \Phi}}{\lambda} \\ \quad + (C_{k, \ell} + D_{k, \ell} \Phi_\ell) \frac{\overline{X}_\ell^{k, u_k, \bar{u}_k, \Phi, \lambda} - \overline{X}_\ell^{k, u_k, \Phi}}{\lambda} w_\ell, \\ \frac{\overline{X}_{k+1}^{k, u_k, \bar{u}_k, \Phi, \lambda} - \overline{X}_{k+1}^{k, u_k, \Phi}}{\lambda} = B_{k, k} \bar{u}_k + D_{k, k} \bar{u}_k w_k, \\ \frac{\overline{X}_k^{k, u_k, \bar{u}_k, \Phi, \lambda} - \overline{X}_k^{k, u_k, \Phi}}{\lambda} = 0, & \ell \in \mathbb{T}_{k+1}. \end{cases}$$

Denote $\frac{\overline{X}_\ell^{k, u_k, \bar{u}_k, \Phi, \lambda} - \overline{X}_\ell^{k, u_k, \Phi}}{\lambda}$ by $Y_k^{k, \bar{u}_k, \Phi}$, which is independent of u_k and λ . Then,

$$\begin{cases} Y_{\ell+1}^{k, \bar{u}_k, \Phi} = (A_{k, \ell} + B_{k, \ell} \Phi_\ell) Y_\ell^{k, \bar{u}_k, \Phi} + (C_{k, \ell} + D_{k, \ell} \Phi_\ell) Y_\ell^{k, \bar{u}_k, \Phi} w_\ell, & \ell \in \mathbb{T}_{k+1}, \\ Y_{k+1}^{k, \bar{u}_k, \Phi} = B_{k, k} \bar{u}_k + D_{k, k} \bar{u}_k w_k, \\ Y_k^{k, \bar{u}_k, \Phi} = 0. \end{cases} \quad (6.12)$$

For any $\ell \in \mathbb{T}_k$, we have

$$\overline{X}_\ell^{k, u_k, \bar{u}_k, \Phi, \lambda} = \overline{X}_\ell^{k, u_k, \Phi} + \lambda Y_\ell^{k, \bar{u}_k, \Phi}. \quad (6.13)$$

Similar to the deviation of (6.5), we have

$$\begin{aligned} d\tilde{J}(k, X_k^{t, x, *}; u_k; \bar{u}_k) &= \lim_{\lambda \downarrow 0} \frac{\tilde{J}(k, X_k^{t, x, *}; u_k + \lambda \bar{u}_k) - \tilde{J}(k, X_k^{t, x, *}; u_k)}{\lambda} \\ &= 2\langle Q_k \overline{X}^{k, u_k, \Phi}, Y^{k, \bar{u}_k, \Phi} \rangle_{\mathbb{T}_k} + 2\langle R_{k, k} (u_k + \Phi_k \overline{X}_k^{k, u_k, \Phi}), \bar{u}_k \rangle_k \\ &\quad + 2\langle R_{k+} (\Phi \overline{X}^{k, u_k, \Phi})|_{\mathbb{T}_{k+1}}, (\Phi Y^{k, \bar{u}_k, \Phi})|_{\mathbb{T}_{k+1}} \rangle_{\mathbb{T}_{k+1}} \\ &\quad + 2\langle G_k \overline{X}_N^{k, u_k, \Phi}, Y_N^{k, \bar{u}_k, \Phi} \rangle_N, \end{aligned} \quad (6.14)$$

and for any $\hat{u}_k \in L_{\mathcal{F}}^2(k; \mathbb{R}^m)$

$$\begin{aligned} d^2 \tilde{J}(k, X_k^{t, x, *}; u_k; \bar{u}_k; \hat{u}_k) &= \lim_{\lambda \downarrow 0} \frac{\tilde{J}(k, X_k^{t, x, *}; u_k + \beta \hat{u}_k; \bar{u}_k) - \tilde{J}(k, X_k^{t, x, *}; u_k; \bar{u}_k)}{\beta} \\ &= 2\langle Q_k \hat{Y}^{k, \hat{u}_k, \Phi}, Y^{k, \bar{u}_k, \Phi} \rangle_{\mathbb{T}_k} + 2\langle R_{k, k} \hat{u}_k, \bar{u}_k \rangle_k + 2\langle R_{k+} (\Phi \hat{Y}^{k, \hat{u}_k, \Phi})|_{\mathbb{T}_{k+1}}, (\Phi Y^{k, \bar{u}_k, \Phi})|_{\mathbb{T}_{k+1}} \rangle_{\mathbb{T}_{k+1}} \\ &\quad + 2\langle G_k \hat{Y}_N^{k, \hat{u}_k, \Phi}, Y_N^{k, \bar{u}_k, \Phi} \rangle_N, \end{aligned}$$

where

$$\begin{cases} \hat{Y}_{\ell+1}^{k, \hat{u}_k, \Phi} = (A_{k, \ell} + B_{k, \ell} \Phi_\ell) \hat{Y}_\ell^{k, \hat{u}_k, \Phi} + (C_{k, \ell} + D_{k, \ell} \Phi_\ell) \hat{Y}_\ell^{k, \hat{u}_k, \Phi} w_\ell, & \ell \in \mathbb{T}_{k+1}, \\ \hat{Y}_{k+1}^{k, \hat{u}_k, \Phi} = B_{k, k} \hat{u}_k + D_{k, k} \hat{u}_k w_k, \\ \hat{Y}_k^{k, \hat{u}_k, \Phi} = 0. \end{cases}$$

If $\hat{u}_k = \bar{u}_k$, then we have

$$d^2 \tilde{J}(k, X_k^{t, x, *}; u_k; \bar{u}_k; \bar{u}_k) = 2\langle Q_k Y^{k, \bar{u}_k, \Phi}, Y^{k, \bar{u}_k, \Phi} \rangle_{\mathbb{T}_k} + 2\langle R_{k, k} \bar{u}_k, \bar{u}_k \rangle_k$$

$$\begin{aligned}
& + 2\langle R_{k+}(\Phi Y^{k, \bar{u}_k, \Phi})|_{\mathbb{T}_{k+1}}, (\Phi Y^{k, \bar{u}_k, \Phi})|_{\mathbb{T}_{k+1}} \rangle_{\mathbb{T}_{k+1}} \\
& + 2\langle G_k Y_N^{k, \bar{u}_k, \Phi}, Y_N^{k, \bar{u}_k, \Phi} \rangle_N,
\end{aligned} \tag{6.15}$$

which is independent of u_k .

Proof of Theorem 3.2. (i) \Rightarrow (ii). Let Φ be a linear feedback equilibrium strategy of Problem (LQ). As noted above, $0|_{\mathbb{T}}$ is an open-loop equilibrium control of the stochastic LQ problem corresponding to (2.1) and (6.10). Hence, for all $\bar{u}_k \in L^2_{\mathcal{F}}(k; \mathbb{R}^m)$ we have

$$d\tilde{J}(k, X_k^{t, x, *}; 0; \bar{u}_k) = 0, \quad d^2\tilde{J}(k, X_k^{t, x, *}; 0; \bar{u}_k; \bar{u}_k) \geq 0. \tag{6.16}$$

Note that $\bar{X}_\ell^{k, u_k, \Phi} = X_\ell^{k, \Phi}$, $\ell \in \mathbb{T}_k$ if $u_k = 0$. Hence, it holds that

$$\begin{aligned}
0 &= \frac{1}{2} d\tilde{J}(k, X_k^{t, x, *}; 0; \bar{u}_k) \\
&= \sum_{\ell=k}^{N-1} \mathbb{E} \left[\left((A_{k, \ell} + B_{k, \ell} \Phi_\ell)^T \mathbb{E}(Z_{\ell+1}^{k, \Phi} | \mathcal{F}_{\ell-1}) + (C_{k, \ell} + D_{k, \ell} \Phi_\ell)^T \mathbb{E}(Z_{\ell+1}^{k, \Phi} w_\ell | \mathcal{F}_{\ell-1}) \right. \right. \\
&\quad \left. \left. + (\Phi_\ell^T R_{k, \ell} \Phi_\ell + Q_{k, \ell}) X_\ell^{k, \Phi} - Z_\ell^{k, \Phi} \right)^T Y_\ell^{k, \bar{u}_k, \Phi} \right] + \mathbb{E} \left[\left(R_{k, k} \Phi_k X_k^{k, \Phi} + B_{k, k}^T \mathbb{E}(Z_{k+1}^{k, \Phi} | \mathcal{F}_{k-1}) \right. \right. \\
&\quad \left. \left. + D_{k, k}^T \mathbb{E}(Z_{k+1}^{k, \Phi} w_k | \mathcal{F}_{k-1}) \right)^T \bar{u}_k \right] \\
&= \mathbb{E} \left[\left(R_{k, k} \Phi_k X_k^{k, \Phi} + B_{k, k}^T \mathbb{E}(Z_{k+1}^{k, \Phi} | \mathcal{F}_{k-1}) + D_{k, k}^T \mathbb{E}(Z_{k+1}^{k, \Phi} w_k | \mathcal{F}_{k-1}) \right)^T \bar{u}_k \right].
\end{aligned} \tag{6.17}$$

As $\bar{u}_k \in L^2_{\mathcal{F}}(k; \mathbb{R}^m)$ can be taken arbitrarily, we can get (3.6). (3.7) follows from (6.15) and (6.16).

(ii) \Rightarrow (i). Reversing the procedure of the above proof, we can achieve the conclusion. The proof is also similar to that of Theorem 2.4. \square